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Dynamics and Chaos: The Spherical Pendulum

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Abstract

All but the simplest of dynamical systems contain nonlinearities that play an important role in modeling and simulating physical systems. They create unpredictable (chaotic) behavior that is often hidden or neglected in traditional solutions. A simple dynamical system, the spherical pendulum, is introduced to illustrate issues, principles, and effects of chaos in dynamics. The spherical pendulum is a two degrees of freedom nonlinear system with a pivot point in space. The equations of motion for the pendulum are derived, simulated, and animated. A periodical perturbation is applied to the pivot point producing radically different behavior.

1. Introduction

Dynamics plays an important role in scientific visualization, especially in simulation and animation. To date, those papers dealing with dynamics in a computer graphics context have assumed linear systems of equations, or linearized solutions of nonlinear systems[1–3]. Most natural phenomena have nonlinear behavior, however; the equations that model them contain nonlinear terms. They are responsible for making a system very sensitive to initial conditions of operation that produce erratic and "unpredictable" behavior. By unpredictable, we mean that one path cannot be inferred from another regardless of how close they were initially. Linear models are inadequate for accurate representation as will be discussed later. This fundamental property called *chaos*, exists only in nonlinear dynamical systems. A two–dimensional clock pendulum is predictable for instance, but perturb its hinge while swinging and it becomes chaotic. Neglecting the nonlinear terms in a dynamical simulation can result in radically different and inaccurate results which cannot be corrected by refinement or greater numerical precision.

The spherical pendulum is a simple, easy to understand dynamical system which we use to demonstrate the principles of chaotic motion and its effects in dynamics. It is a two degrees of freedom pendulum with a pivot point in space, that is, it is a weight on the end of a rigid shaft which pivots about a point in space. The pivot point may be subject to perturbations. Such a system is difficult to realize physically; the motion of the shaft interferes with any solid structure attaching to the pivot point.

There are important principles pertaining to accurate and realistic dynamics that the spherical pendulum demonstrates. Using the example of the spherical pendulum, we will illustrate the following:

(1) Basics of (chaotic) dynamical systems.
(2) Methods of setting up equations.
(3) Numerical methods for solving the equations.
(4) Characteristics of chaotic motion.

Point (2), and to a lesser extent (3), are application dependent, but some general observations will be abstracted from them. Section 2 defines terms and introduces basic concepts from dynamical systems. Section 3 develops the equations for the spherical pendulum. Section 4 discusses the numerical solution of the equations and characterizes the results. General principles of chaotic motion are drawn from these results.

2. Dynamical Systems Concepts

Definitions. A *dynamical system* is any physical, biological, etc. phenomenon that changes in time, or alternatively, a mathematical model that describes the evolution of a system in time. If time is assumed to be continuous then the dynamical system is represented by $\frac{dx}{dt} = F(x(t))$, $x \epsilon R^n$, where $x(t)$ is a set of coordinates describing the system, and F is a single-valued function describing the evolution of the state $x(t)$ at time t. Continuous dynamical systems are also called *flows* and in this sense $F(x(t))$ describes the direction of the flow at time t and state $x(t)$. Taken at discrete intervals the dynamical system is represented by $x(t_{n+1}) = F(x(t_n))$, or simply $x_{n+1} = F(x_n)$. Discrete dynamical systems are also called *maps* and in this sense $F(x(t_n))$ maps the system from state $x(t_n)$ at time t_n to state $x(t_{n+1})$ at time t_{n+1}. The spherical pendulum and a cellular automata are examples of continuous and discrete systems, respectively.

A *linear system* is one for which the principle of superposition holds. It indicates that given any initial states $x(t_1)$ and $x(t_2)$ the flow (or map) F satisfies: $F(x(t_1) + x(t_2)) = F(x(t_1)) + F(x(t_2))$. A *nonlinear system* $\frac{dx}{dt} = F(x(t))$ can be *linearized* about a point $x(t_c)$ by redefining the system as $\frac{dy}{dt} = J(y(t))$, where J is the Jacobian of F evaluated at $x(t_c)$. The new system is useful to describe the dynamics near the point of linearization $x(t_c)$. Given an initial state $x(t_0)$ and time $t_1 \neq t_0$, a system is *deterministic* or *predictable* if the state at time t_1, $x(t_1)$, can be uniquely determined, i.e. the trajectories or flows of a deterministic dynamical system do not intersect or split apart in time.

A *forward trajectory (orbit or path)* starting at state $x(t_0)$ is defined by the recurrent relation: $F(F(...F(x(t_0))...)) = F^n(x(t_0))$, where n is any nonnegative integer. *Backward trajectories* are similarly defined using the inverse of the flow F^{-1}. Deterministic continuous flows are always invertible but this is not the case for discrete systems, except for those obtained from continuous flows.

A trajectory of a continuous system is said to be *periodic* with period t_p if $F(x(t_0)) = F(x(t_0 + t_p))$. When $F(x(t_0)) = x(t_0)$ then $x(t_0)$ is called a *fixed point* or *equilibrium point* of the flow. Similar definitions follow for discrete systems.

The *Lyapunov exponents* of a dynamical system measure the long-term evolution of an infinitesimal n-sphere (actually an n-ellipsoid due to the locally deforming nature of the flow) of initial conditions subject to the flow of the system, and they are defined as :

$$\lambda_i = \lim_{t\to\infty} \lim_{\epsilon\to 0} \frac{1}{t} \frac{\epsilon_i(t)}{\epsilon(0)}$$

where $\epsilon(0)$ is the radius of the infinitesimal n-sphere, centered on a point at state $x(0)$. As time evolves the flow deforms the n-sphere and thus, $\epsilon_i(t)$ is the length of the principal axis of the ellipsoid (deformed n-sphere) at time t.

The trajectories of an n-dimensional state space have n Lyapunov exponents, called the *Lyapunov spectrum*, and they are related to the expanding or contracting nature of different directions in phase space. We use the sign of the Lyapunov exponents to characterize the behavior of a dynamical system. A positive (negative) Lyapunov exponent indicates exponential expansion (contraction) of nearby trajectories.

When the motion is unbounded, a positive Lyapunov exponent indicates that nearby trajectories diverge and eventually go to infinity, this is typical of linear systems. But, when the motion is bounded in some region, trajectories cannot go to infinity and a positive Lyapunov exponent makes no sense unless widely separated trajectories are eventually folded back together and thus contained in the bounded region. This folding and stretching of trajectories causes the long–term behavior of an initial condition to be unpredictable and is called *Chaos*. "Chaos is then caused by local stretching (separation of nearby trajectories) and, global folding (bounded motion)."

Systems that conserve energy are called *conservative*, otherwise they are called *dissipative*. A pendulum can be modeled either as a conservative (no damping) or as a dissipative (with damping) system. Conservative systems can be modeled using Hamilton's equations: $-\frac{\partial p_i}{\partial t} = \frac{\partial H}{\partial q_i}$ and $\frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i}$, where p_i and q_i are the canonical conjugate state variables, and the *Hamiltonian H* is related to the energy and is a constant.

Prior to 1961 the following principle was generally accepted: "The forward trajectories of two initial states $x(t_0)$ and $x(t_1)$ can be made arbitrarily close by choosing the initial states sufficiently close." In 1961, Edward Lorenz's weather model produced patterns that diverged unpredictably from two *almost* indistinguishable initial conditions[4]. This property, now known as *sensitive dependence on initial conditions*, became the pillar of the theory of chaos. Chaotic systems exponentially separate trajectories. **This property can only be found in nonlinear systems; it is impossible to reproduce it even with the best linear model**.

Chaotic systems not only separate trajectories, but they also fold them back into the same region; thus chaotic motion coexists with bounded motion. This stretching and folding usually results in very complicated self-similar structures, i.e. fractal sets, and in the creation of periodic trajectories winding densely into the same region. Surprisingly, erratic and unpredictable motion coexists with predictable and deterministic motion.

Some systems exhibit transient behavior before settling down into their typical behavior, oftently represented by an attractor. An *attractor*, roughly speaking, is a region where nearby trajectories are attracted. A dynamical system might possess different attractors. For more details see Eubank and Farmer[5] and Wiggins[6].

3. Modeling the Equations of Motion

A simple one–degree of freedom damped pendulum is known to exhibit complex dynamics, and to behave chaotically under the influence of small perturbations. The dynamics of a *spherical pendulum* are more complicated. Because the motion takes place on a sphere, we use spherical coordinates to derive the equations of motion. A point P is defined by a triple $P(r, \theta, \varphi)$, where r is the distance from P to the origin, θ is the angle defined by the segment $OP(O = origin)$ with the *z–axis*, and φ is the angle defined by the projection of OP onto the *xy–plane* with the *x–axis*.

3.1. The Unperturbed Problem

Consider an undamped pendulum of mass m and rigid chord of length l attached to the origin of the coordinate system. The only force involved is gravity. Only two variables, the angles θ and φ, are needed to specify its position, making it a two–degrees of freedom system. The position of the pendulum with respect to the Cartesian coordinate system is specified by the spherical transformation: $x = l \sin\theta \cos\varphi$, $y = l \sin\theta \sin\varphi$, and $z = l\cos\theta$.

The dynamics of motion can be described by the Lagrangian $L = T - V$, where T and V represent the kinetic and potential energies, respectively. The kinetic energy T is always given by $T = \frac{1}{2}mv^2$, where v is the velocity of the pendulum †. The potential energy V can be modeled in different ways according to the system. We assume the pendulum reaches a maximum potential energy when $\theta = 0$ (north pole), and a minimum potential energy when $\theta = \pi$, so we can write: $V = mgz$. Note that V can be negative. (If V is required to be nonnegative, an alternative is $V = mg(l + z)$). Using spherical coordinates we get:

$$T = \frac{1}{2}m\left[\left(l\dot{\theta}\right)^2 + (l\sin\theta\dot{\varphi})^2\right] \quad (1)$$

$$V = mgl\cos\theta \quad (2)$$

where $\dot{\theta} = \frac{d\theta}{dt}$ and $\dot{\varphi} = \frac{d\varphi}{dt}$. Substituting (1) and (2) into the Lagrangian we obtain:

$$L = \frac{1}{2}m\left[\left(l\dot{\theta}\right)^2 + (l\sin\theta\dot{\varphi})^2\right] - mgl\cos\theta \quad (3)$$

Hamiltonian Equations of Motion. The Hamiltonian H is defined by: $H = T + V$. Since T and V are kinetic and potential energy, respectively, then H represents the total energy of the system. Using the Lagrangian we introduce two variables:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \quad (4)$$

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = ml^2\sin^2\theta\dot{\varphi}$$

where p_θ and p_φ describe the angular momentum of the pendulum with respect to θ and φ, respectively. From these last two equations we obtain, after substitution into (1), the Hamiltonian:

$$H = \frac{1}{2m}\left[\frac{P_\theta^2}{l^2} + \frac{P_\varphi^2}{l^2\sin^2\theta}\right] + mgl\cos\theta$$

Equation (5) quantifies the energy of the pendulum and is pivotal in deriving the equations of motion. Some singularities are present in this equation and are described together with other cases in Section 4. The only property mentioned at this point is that $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$ which indicates that energy is conserved, as expected. The quations of motion are obtained using the properties of the Hamiltonian H (see Section 2), that is, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, and $-\dot{p}_i = \frac{\partial H}{\partial q_i}$, where, in our problem, we have: $q_1 = \theta, q_2 = \varphi, p_1 = p_\theta$, and $p_2 = p_\varphi$. Applying these conditions we derive:

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{ml^2} \\ \dot{p_\theta} &= \frac{\cos\theta}{ml^2\sin^3\theta}p_\varphi^2 + mgl\sin\theta \\ \dot{\varphi} &= \frac{p_\varphi}{ml^2\sin^2\theta} \\ \dot{p_\varphi} &= 0 \end{aligned} \quad (6)$$

This equations represent the motion of an undamped spherical pendulum; when the pendulum starts moving with energy H it will never stop. In practice, however, one would expect some sort of friction (probably due to air) to slow down the motion of the pendulum. So, if the pendulum is damped with damping coefficients δ_1 and δ_2, acting on the angular velocities $\dot{\theta}$ and $\dot{\varphi}$, respectively, then the previous system of equations becomes:

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{ml^2} \\ \dot{p_\theta} &= \frac{\cos\theta}{ml^2\sin^3\theta}p_\varphi^2 + mgl\sin\theta - \delta_1\frac{p_\theta}{ml^2} \\ \dot{\varphi} &= \frac{p_\varphi}{ml^2\sin^2\theta} \\ \dot{p_\varphi} &= -\delta_2\frac{p_\varphi}{ml^2\sin^2\theta} \end{aligned} \quad (7)$$

3.2. The Periodically Perturbed Problem

Now assume that a periodical perturbation is applied to the pivot point of the pendulum in the direction of the *z–axis*. The perturbation is defined by $\epsilon\sin(\omega t)$, where ϵ is the amplitude of the periodical force and ω is its frequency. The potential energy V is now $V = mgl(1 -$

† The velocity can be expressed as $v = \dot{x} + \dot{y} + \dot{z}$, where implies derivation with respect to t.

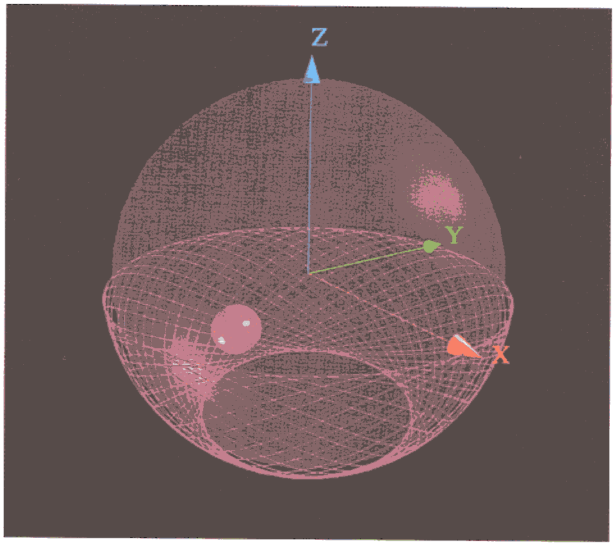
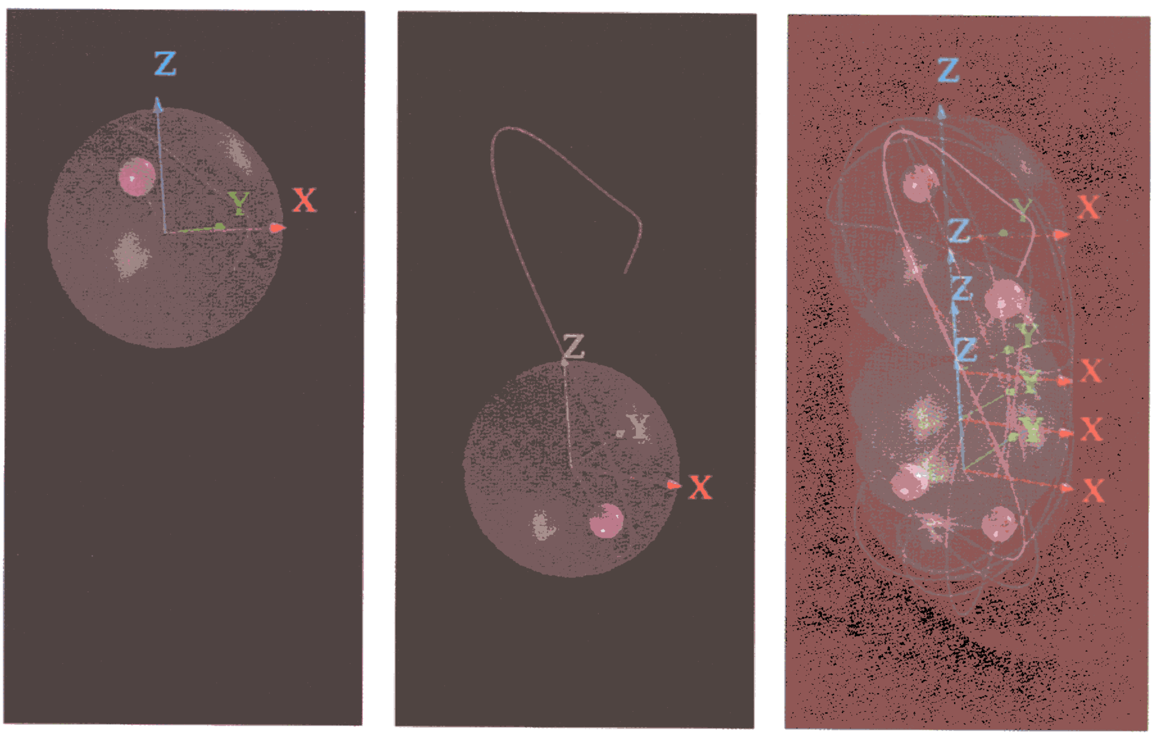
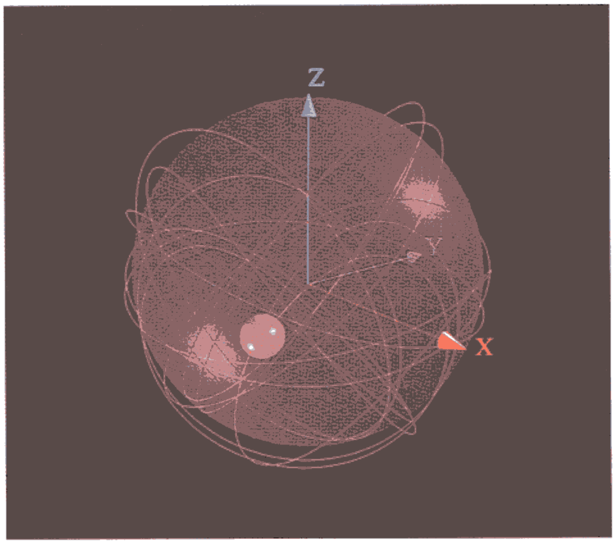


Figure 1: *3D rendering of a spherical pendulum.*



(a) initial motion (b) trajectory after a few seconds (c) overlaid motion

Figure 5: *Sequence of snapshots illustrating the chaotic motion of a perturbed spherical pendulum.*



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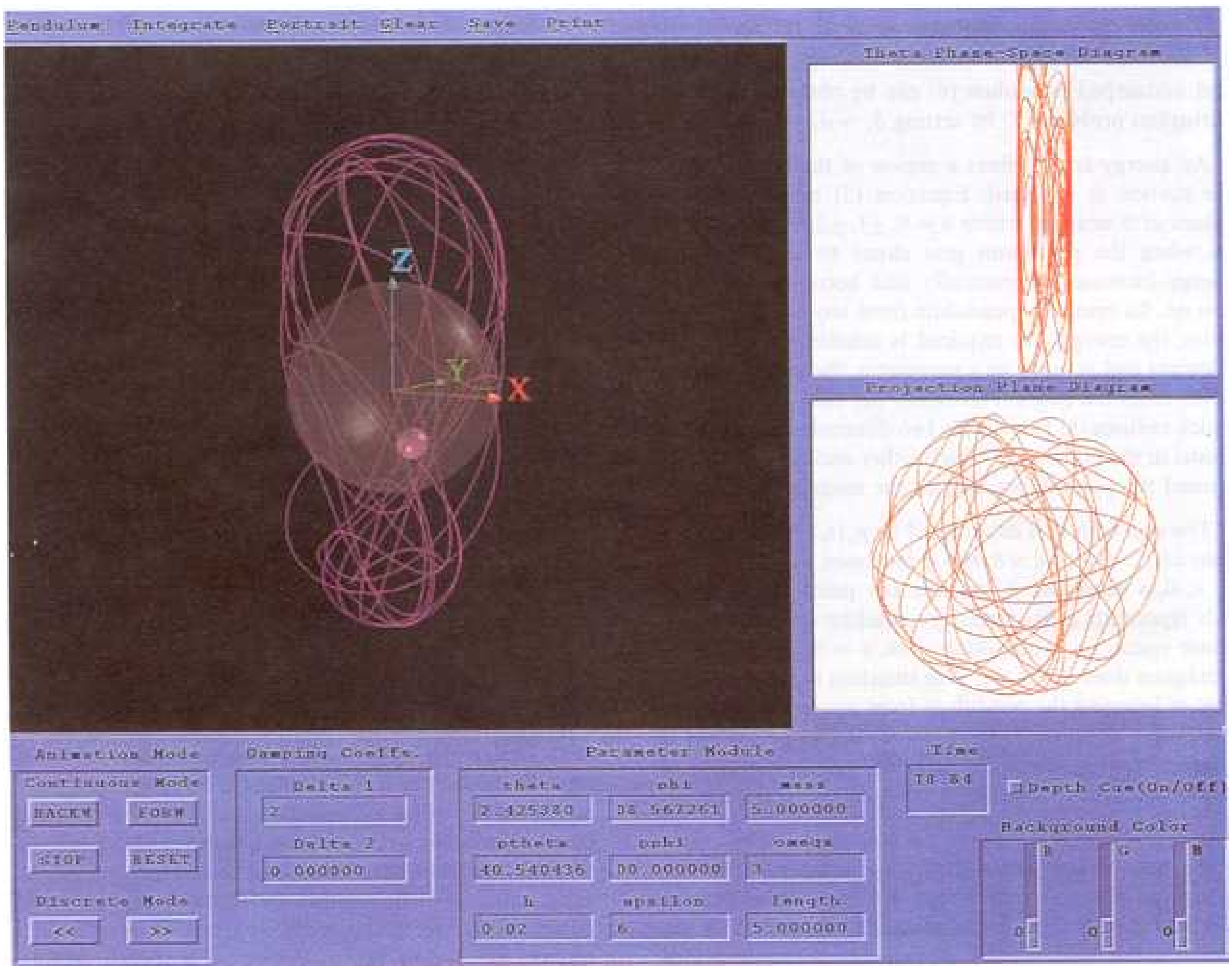

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$\epsilon \sin(\omega t)) \cos\theta$. A similar derivation of the Hamiltonian equations, after suspending the system by introducing a new variable $\gamma = \omega t$, produces:

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{ml^2} \\ \dot{p_\theta} &= \frac{\cos\theta}{ml^2 \sin^3\theta} p_\varphi^2 + mgl(1 - \epsilon \sin\gamma)\sin\theta - \delta_1 \frac{p_\theta}{ml^2} \\ \dot{\varphi} &= \frac{p_\varphi}{ml^2 \sin^2\theta} \\ \dot{p_\varphi} &= -\delta_2 \frac{p_\varphi}{ml^2 \sin^2\theta} \end{aligned} \quad (8)$$

This is the system of equations we will be solving to simulate the motion of the pendulum. Other forms of the equations of motion may also be obtained, but we chose this form for practical purposes mentioned in the next section.

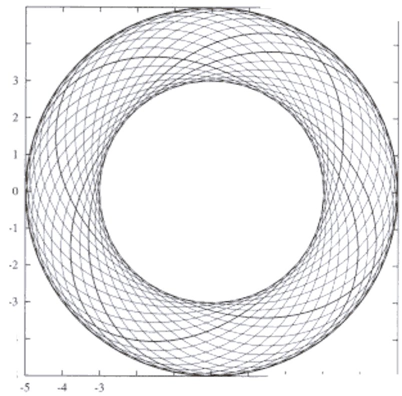

Figure 2: *Forward trajectory of an unperturbed spherical pendulum projected on the xy–plane.*

4. Simulation and Animation

To simulate the motion of the pendulum, equation (8) is solved numerically for the position (l, $\theta(t)$, $\varphi(t)$) as functions of time t. Observe that the unperturbed and undamped pendulum (6) can be obtained from the perturbed problem (8) by setting $\delta_1 = \delta_2 = \epsilon = 0$.

An energy level defines a region of the sphere where the motion is confined. Equation (5) reveals that for values of θ near $n\pi$ where $n = 0, \pm1, \pm2, ...$, and $p_\varphi \neq 0$, i.e. when the pendulum gets closer to the poles, the energy increases dramatically and becomes infinite at $\theta = n\pi$. To bring the pendulum from any position to the poles, the energy (H) required is infinite. Because H is constant and is input as a parameter, the pendulum will never cross the poles. This is not the case when $p_\varphi = 0$, which reduces (8) to a simple two dimensional pendulum model in the plane $\varphi(t_0)$ which either oscillates or rotates around the poles depending on the magnitude of p_θ.

The sign of $\dot{\varphi}(t)$ is determined by $p_\varphi(t_0)$. When $p_\varphi > 0$, then $\dot{\varphi}(t) > 0$ for all $t \epsilon R$ and φ increases. Similarly when $p_\varphi < 0, \varphi$ decreases. Note that any point along the *z–axis* represents a fixed point or equilibrium point of the phase space, i.e. when $\theta(t_0) = n\pi, n = 0, \pm1, \pm2, ...$, the pendulum does not move. This situation is different from that of bringing the pendulum from another position to the poles, in which case $H = \infty$. If the pendulum is already moving, then we need an infinite amount of energy to make it spin about the *z–axis*, but it does not move if placed there initially.

We have solved (8) numerically using a Predictor–Corrector method[7] with Runge–Kutta of order four as the predictor. Runge–Kutta methods slowly destroy the Hamiltonian structure by introducing dissipative artifacts. They are negligible for short periods of time, but become significant with increasing time intervals (several days in our experiment). The following cases (assuming $m = 5.0, g = 32.2, l = 5.0$) are presented.

Case 1. Three dimensional unperturbed ($\epsilon = 0$) and undamped ($\delta_1 = \delta 2 = 0$) pendulum with $\theta(0) = \frac{\pi}{2}, \varphi(0) = 0, p_\theta(0) = 10, p_\varphi(0) = 300$. Figure 1 depicts a three dimensional rendering of the pendulum. An outer sphere is used to visualize the pendulum's forward trajectory. The projection of this trajectory on the *xy–plane* is shown in Figure 2. As $p_\varphi(0)$ increases, the pendulum's trajectory tends to be planar. (When $p_\varphi \gg p_\theta$, the angular momentum p_φ dominates the motion). Similarly when $p_\varphi \ll p_\theta$, the pendulum is more likely to move near the poles. Recall that H, or equivalently p_θ, must be infinite to make the pendulum spin on the *z–axis*. Different combinations of p_θ and p_φ result in different regions of motion, but their projection on the *xy–plane* is always the same: a well defined annulus.

Case 2. A periodic force of the form $\epsilon sin(\omega t)$ is applied along the *z–axis*. The amplitude is set to $\epsilon = 6.0$, with frequency $\omega = 3.0$, damping coefficients: $\delta_1 = 2.0, \delta_2 = 0.0$ and $\theta(0) = 0.6283, \varphi(0) = 0.0, p_\theta(0) = 4.0, p_\varphi(0) = 300.0$. Figure 3 shows, after eliminating transient behavior, the chaotic attractor for the θ phase–space portrait of motion. It is a very complicated diagram and indicates that the pendulum is now moving erratically. Using the algorithm described by Benettin et al.[8] we found the Lyapunov spectrum: $(+, -, -, 0, 0)$, indicating the existence of a Chaotic Attractor. The positive exponent indicates exponential spreading within the attractor and negative exponents indicate exponential contraction onto the attractor. We follow the trajectory of a second nearby initial condition: $\theta_0(0) = 0.6285, \varphi_0(0) = 0.001$ (all other

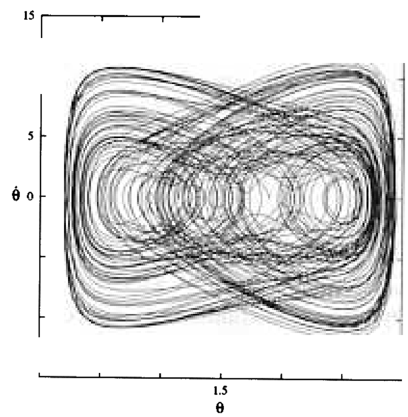

Figure 3: *Chaotic attractor in θ phase space.*

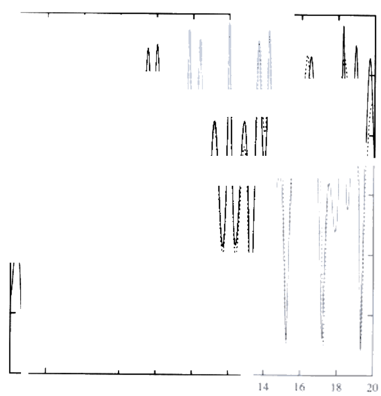

Figure 4: *Divergence of two nearby θ(t) orbits due to sensitive dependence on initial conditions.*

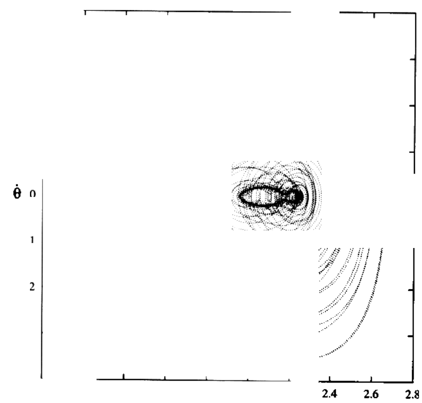

Figure 6: *A second chaotic attractor in θ(t) phase space.*

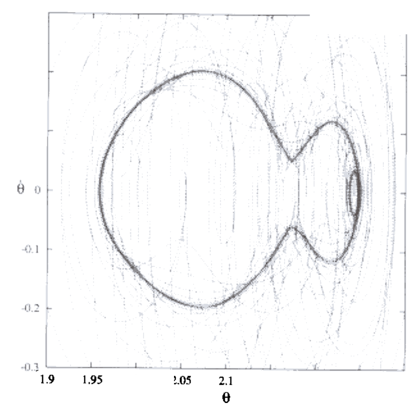

Figure 7: *Magnification of Figure 6. A periodic orbit is found embedded in the chaotic attractor.*

parameters are kept the same). At first, the two inital conditions generate close trajectories, but they eventually diverge indicating sensitive dependence on initial conditions, as shown in Figure 4. Figure 5 shows a progressive sequence of three snapshots, taken at different time intervals, indicating the path traced by the pendulum as its attachment point is periodically perturbed.

Case 3. We perturb the pendulum with amplitude $\epsilon = 0.5$, frequency $\omega = 1.0$, damping coefficients: $\delta_1 = 10.0, \delta_2 = 0.0$ and initial conditions: $\theta(0) = 0.6283, \varphi(0) = 0.6283, p_\theta(0) = 10.0, p_\varphi(0) = 300.0$. Computation of the Lyapunov spectrum: $(+,-,-,0,0)$ indicates the existence of another chaotic attractor, shown in Figure 6. After magnification we find a periodic trajectory embedded in the attractor. The small magnitude of the positive exponent (approximately $\lambda_1 = 0.022$) indicates that the periodic orbit is not quite stable. Initial conditions near the periodic orbit are still subject to chaotic motion as one can observe on Figure 7. The trace of the trajectory corresponding to the first initial condition is shown in Figure 8. We note that the projection is still bounded by an annulus, but it is no longer deterministic as in Case 2. A linear vs. nonlinear solution is shown in Figure 9. They are initially close to each other, but soon separate.

We developed a graphical user interface to facilitate the visualization of the dynamics of the spherical pendulum. This interface, shown in Figure 10, allows the interactive modification of most of the parameters which affect the pendulum's motion. The large graphics window presents a three dimensional continuous animation of the pendulum's path. The window in the upper right corner of the interface shows the current θ phase–space (angular position vs. angular velocity) plot. Finally, a

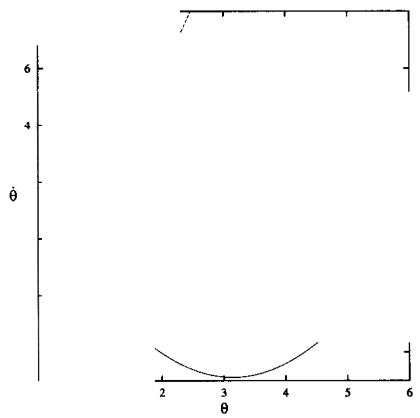

Figure 9: *Linear vs. nonlinear solutions.*

projection onto the *xy–plane* of the pendulum's trajectory is displayed in the lower window.

5. Conclusions

A common practice when simulating and animating natural phenomena using computer graphics is to reproduce motion that is realistic. With regard to this problem, we have discussed:

- Nonlinear dynamical systems are capable of producing radically different patterns from two almost indistinguishable initial conditions—a fundamental property of chaos known as *sensitive dependence* on initial conditions.
- When setting up the equations of motion of a system there may be different models. However, sensitive dependence on initial conditions may only be found in those models with nonlinear terms. Chaos is a nonlinear phenomenon.
- Simulating and animating a dynamical system is usually obtained by integrating the equations of motion that describe it. Numerical techniques sometimes introduce artifacts like round–off errors or instability, and should be considered when testing for sensitive dependence on initial conditions. They may also destroy some properties of a system like conservation of energy. **Linear solutions are intrinsically ill–suited for chaotic motion.**
- Chaotic systems make nearby trajectories diverge exponentially, and keep the motion bounded by globally folding the trajectories; small errors are quickly amplified, making the system unpredictable. Periodic orbits are embedded inside chaotic regions.

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