

Cycling Behavior in Near-Identical Cell Systems.

Antonio Palacios, Patrick Longhini
Department of Mathematics
San Diego State University
San Diego, CA 92182-7720
USA

June 12, 2002

Abstract

A generic pattern of collective behavior of symmetric networks of coupled identical cells is cycling behavior. In networks modeled by symmetric systems of differential (difference) equations, cycling behavior appears in the form of solution trajectories (orbits) that linger around symmetrically related steady-states (fixed points) or periodic solutions (orbits) or even chaotic attractors. In this last case, leading to what is called “cycling chaos”. In particular, Dellnitz *et al.* [5] demonstrated the existence of cycling chaos in continuous-time three-cell systems modeled by Chua’s circuit equations and Lorenz equations, while Palacios [24, 25] later demonstrated the existence of cycling chaos in discrete-time cell systems. In this work, we consider two issues that follow-up from these previous works. First of all, we address the generalization of existence of cycling behavior in continuous-time systems with more than three cells. We demonstrate that increasing the number of cells, while maintaining the same network connectivity used by Dellnitz *et al.* [5], is not enough to sustain the nature of a cycle, in which only one cell is active at any given time. Secondly, we address the existence of cycling behavior in networks with near-identical cells, where the internal dynamics of each cell is governed by an identical model equation but with possibly different parameter values. We show that, under a new connectivity scheme, cycling behavior can also occur in networks with near-identical cells.

1 Introduction

A wide variety of complex spatio-temporal phenomena are commonly modeled by coupled cell systems through either continuous-time systems of differential equations or discrete-time systems of difference equations. Some examples include: the dynamics of arrays of Josephson junctions [2, 8, 14], central pattern generators in biological systems [15, 16, 28], laser

dynamics [27, 31], synchronization of chaotic oscillators [26, 32], the dynamics of competing species [20, 21, 22], collective behavior of bubbles in fluidization [13], and the flocking of birds [30]. In these examples, three factors are normally considered when studying the collective behavior that a particular network can produce. Mainly, the internal dynamics of each individual cell, the topology of cell connections (i.e., which cells communicate with each other), and the type of coupling. In recent years, however, symmetry considerations have also gained attention. More details can be found in the work by Dionne *et al.* [6, 7], Golubitsky and Stewart [12]. For instance, one particular pattern of behavior that is commonly found in symmetric systems of coupled identical cells is cycling behavior, in which solution trajectories can linger around steady states and periodic solutions for increasingly longer periods of time. These type of cycles are formally called *heteroclinic* if the solutions that are part of the cycle are all different. Otherwise the cycles are called *homoclinic*.

Dellnitz *et al.* [5] have shown that symmetric identical cell systems can also produce cycling behavior that is independent of the internal dynamics of each individual cell. Using Chua’s circuit equations and Lorenz equations, Dellnitz and collaborators further illustrate this conclusion with simulations of a network of three identical cells connected in a directed ring fashion. In these simulations, solution trajectories can cycle around symmetrically related chaotic sets. Thus producing “cycling chaos”. In more recent work, we showed, first numerically [24] and then analytically [25], that cycling chaos also occurs in symmetric systems of coupled identical cells described by discrete-time maps. Two issues that arise from these works include: the generalization of cycling behavior in systems of coupled identical cells with more than three cells; and the existence of cycling behavior in near-identical cells. By “near-identical” we mean cells whose internal dynamics is governed by identical equations but with possibly different parameter values.

In this work, we address these two issues over cell systems modeled by continuous-time differential equations and discrete-time maps as well. In particular, we show that increasing the number of cells in the type of network employed by Dellnitz *et al.*, i.e., a directed ring with \mathbf{Z}_n symmetry, will destroy (even after adjusting other parameters such as coupling strength) the type of cycling behavior in which only one cell is active at any given time, while the others are quiescent. Adding an all-to-all coupling to non-nearest neighboring cells, however, can restore the desired cycle. We then use this type of network connectivity to show the existence of cycles in near-identical cells. This latter type of cycles allows for trajectories that connect a wider range of solutions, including steady-states (fixed points), periodic solutions (periodic orbits), and chaotic attractors—all in the same trajectory. Various computer simulations are used to illustrate the main results.

2 Background

2.1 Heteroclinic Cycles

A *heteroclinic cycle* is a collection of solution trajectories that connects sequences of equilibria and/or periodic solutions of continuous or discrete systems. As time evolves, a typical

nearby trajectory stays for increasingly longer periods near each solution before it makes a rapid excursion to the next solution. For a more precise description of heteroclinic cycles and their stability, see Melbourne *et al.* [23], Krupa and Melbourne [18], the monograph by Field [10], and the survey article by Krupa [17]. The existence of structurally stable heteroclinic cycles is considered a highly degenerate feature of both types of systems, continuous and discrete. In other words, typically they do not exist. In continuous systems, where the governing equations normally consist of systems of differential equations, it is well-known that the presence of symmetry can lead, however, to structurally stable, asymptotically stable, cycles [9, 11]. First, symmetry forces certain subspaces of the phase-space to be invariant under the governing equations. Then, cycles are formed through saddle-sink connections between equilibria and/or periodic solutions that lie on the invariant subspaces. Since saddle-sink connections are structurally stable so are the cycles. Figure 1 shows an example of a cycle involving three steady-states of a system of ODE's proposed by Guckenheimer and Holmes [11]. Observe that, as time evolves, a nearby trajectory stays longer on each equilibrium.

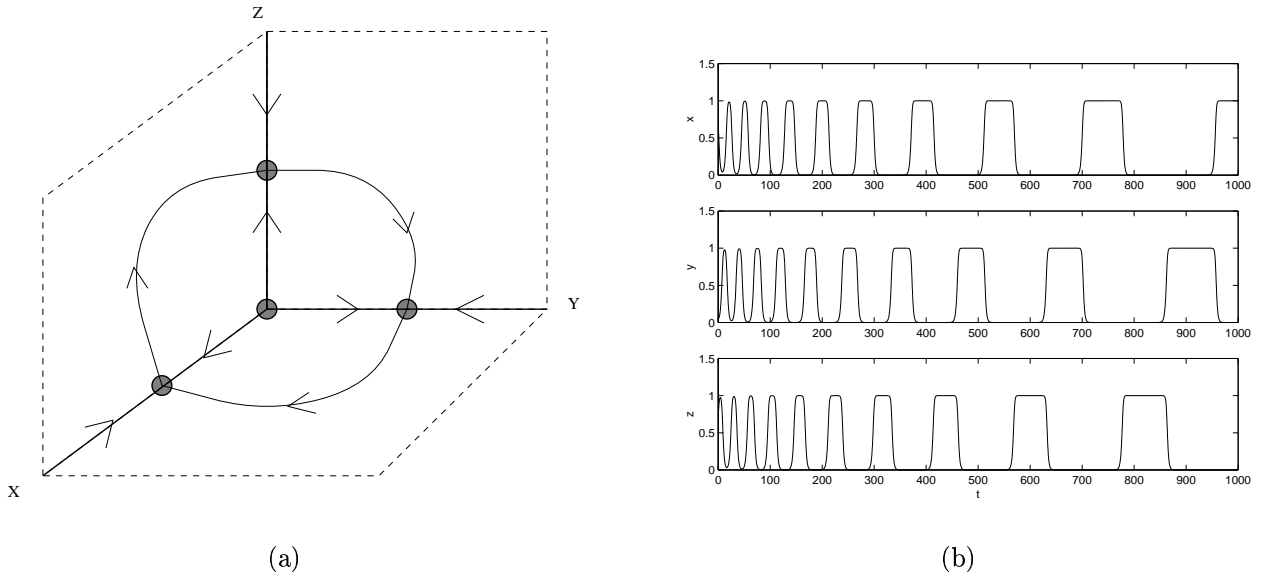


Figure 1: Heteroclinic cycle connecting three equilibrium points of the Guckenheimer and Holmes system. (a) Saddle-sink connections in phase-space, (b) Time series evolution of a typical nearby trajectory.

For systems whose symmetries are described by the continuous group $\mathbf{O}(2)$, i.e. the group of rotations and reflections on the plane, Armbruster *et al.* [1] show that heteroclinic cycles between steady-states can occur stably, and Melbourne *et al.* [23] provide a method for finding cycles that involve steady-states as well as periodic solutions. For systems with discrete symmetries, in particular Dihedral \mathbf{D}_n symmetry, Buono *et al.* [3] show that cycles connecting steady-states and periodic solutions are also found stably in systems of coupled identical cells.

Regardless of the type of symmetry, when a heteroclinic cycle is also asymptotically stable, it can serve as a model for a certain kind of intermittency, since nearby trajectories move quickly between solutions and stay for a relatively long time near each solution.

2.2 Coupled Cell Systems of Differential Equations

In this section, we revisit symmetric systems of coupled identical cells similar to those employed by Dellnitz *et al.* [5] in the study of cycling chaos. In particular, we consider systems with N identical cells, where the internal dynamics of each cell is governed by a k -dimensional continuous-time system of differential equations of the form

$$\frac{dX_i}{dt} = f(X_i, \lambda), \quad (1)$$

where $X_i = (x_{i1}, \dots, x_{ik}) \in \mathbf{R}^k$ denotes the state variables of cell i and $\lambda = (\lambda_1, \dots, \lambda_p)$ is a vector of parameters. Observe that f is independent of i because the cells are assumed to be identical. A network of N cells is a collection of identical interconnected cells, which we model by the following system of coupled differential equations

$$\frac{dX_i}{dt} = f(X_i) + \sum_{j \rightarrow i} c_{ij} h(X_i, X_j), \quad (2)$$

where h is the coupling function between two cells, the summation is taken over those cells j that are coupled to cell i , and c_{ij} is a matrix of coupling strengths.

2.3 Local and Global Symmetries

Following Dellnitz *et al.* [5], we distinguish *local* symmetries from *global* symmetries. $\mathcal{L} \subset \mathbf{O}(k)$ is the group of local or internal symmetries of individual cells if, for all $l \in \mathcal{L}$, we have

$$f(l X_i) = l f(X_i).$$

While local symmetries are dictated by f , global symmetries are induced by the pattern of coupling. More precisely, $\mathcal{G} \subset \mathbf{O}(N)$ is the group of global symmetries of the network if, for all $\sigma \in \mathcal{G}$, we have

$$F(\sigma X) = \sigma F(X).$$

Depending on the coupling function h , it is possible for the local symmetries l to be also symmetries of the network equations (2). In particular, when the action of l on each cell individually is a symmetry of (2), so that

$$\begin{aligned} h(X_i, l X_j) &= l h(X_i, X_j) \\ h(l X_i, X_j) &= h(X_i, X_j), \end{aligned}$$

for all $l \in \mathcal{L}$, then the coupling is called *wreath product* coupling.

2.4 Cycling Chaos

In principle, saddle-sink connections can also lead to more “complicated” cycling behavior involving other type of solutions besides steady-states. The only requirement is for the appropriate invariant subspaces to contain those solutions. For instance, replacing the equilibria in the Guckenheimer-Holmes system with chaotic attractors can lead to what Dellnitz *et al.* [5] call *cycling chaos*. They do this as follows. First, Golubitsky *et al.* [5] show that the Guckenheimer-Holmes system can be interpreted as a network of three identical coupled cells. Dellnitz *et al.* [5] then made the critical observation that, under certain conditions, cycling behavior is a feature of the global dynamics that can persist independently of the internal dynamics of each cell. It follows that if the internal dynamics of the Guckenheimer-Holmes system is replaced by a chaotic attractor, then the new coupled system would produce cycling chaos. Dellnitz *et al.* demonstrate this conclusion using first a modified version of Chua’s circuit equations

$$\begin{aligned}\dot{x}_{i1} &= \alpha(x_{i2} - m_0x_{i1} - \frac{m_1}{3}x_{i1}^3) \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3} \\ \dot{x}_{i3} &= -\beta x_{i2},\end{aligned}\tag{3}$$

where i denotes the cell number, α , m_0 , m_1 and β are parameters; and then Lorenz equations

$$\begin{aligned}\dot{x}_{i1} &= \sigma(x_{i2} - x_{i1}) \\ \dot{x}_{i2} &= \rho x_{i1} - x_{i2} - x_{i1}x_{i3} \\ \dot{x}_{i3} &= x_{i1}x_{i2} - \beta x_{i3}.\end{aligned}\tag{4}$$

where σ , ρ and β are parameters. Both cell equations possess local reflectional symmetry. More precisely, Chua’s equations are symmetric under the substitution $(x_{i1}, x_{i2}, x_{i3}) \mapsto (-x_{i1}, -x_{i2}, -x_{i3})$, while Lorenz equations are symmetric under the substitution $(x_{i1}, x_{i2}) \mapsto (-x_{i1}, -x_{i2})$. Figure 2 illustrates the results of simulations with these last two equations. Observe that only one cell is active at any given time.

2.5 Networks with $N > 3$ Identical Cells

We now address the question of whether a network of $N > 3$ identical cells interconnected in a directed ring fashion can also produce cycling chaos. We consider this question with a wreath coupling function similar to that employed by Dellnitz *et al.* [5]. That is, a coupling of the form $h(X_i, X_j) = -\gamma\|X_j\|X_i$, with identical coupling strength $\gamma > 0$. Through numerical work, we have found that a directed ring of $N > 3$ identical cells will not necessarily produce cycling behavior, even after adjusting the coupling strength. We have also found, however, that adding an all-to-all coupling scheme (see Figure 3) between non-nearest neighboring cells can help preserve the cycling behavior. Figure 4 illustrates this conclusion in a network with 5 cells, where the internal dynamics of each individual cell is also modeled by Chua’s equation (3). Observe in Figure 4(a) that when the cells are connected in a directed ring fashion only, as is the case of those cells used in Figure 2, the nature of the cycle, in which only one cell is active at any given time disappears. Interestingly, increasing the number of

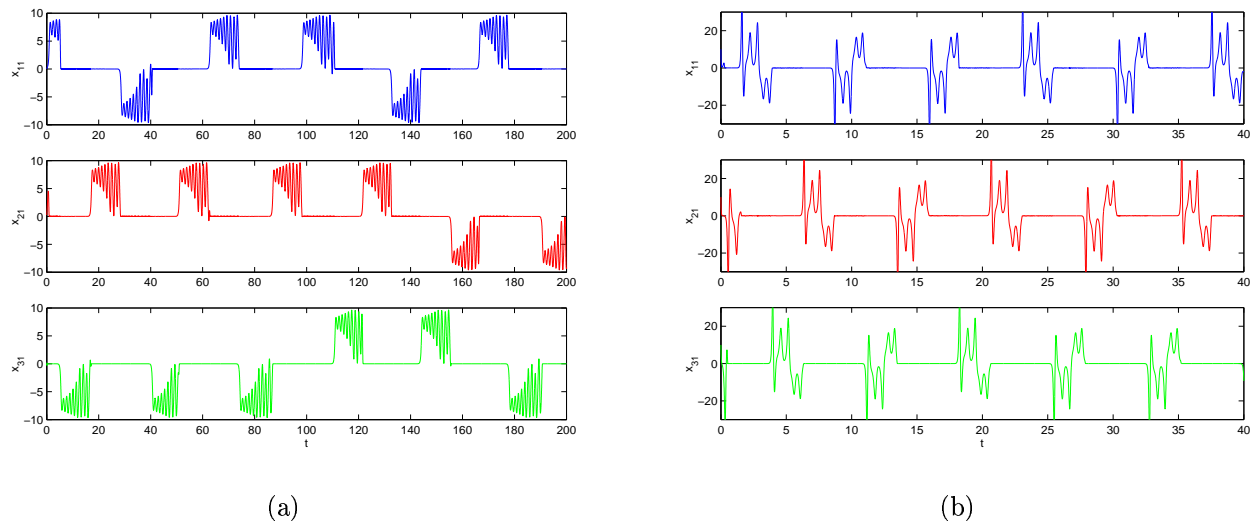


Figure 2: Cycling chaos in a network of three identical cells connected in a directed ring fashion, discovered by Dellnitz *et al.* [5]. The internal dynamics of each cell is modeled by (a) Chua’s circuit equations and (b) Lorenz equations.

cells, does not destroy the intermittent behavior of the dynamics of each cell. The persistence of this intermittency can be explained by recalling the fact that saddle-sink connections are structurally stable. Consequently, a perturbation of the network such as increasing the number of cells (without adding any further connections) can break saddle-sink connections in a cycle but the overall “intermittency” nature of the cycle would persist. When all-to-all coupling is added to non-nearest neighboring cells, only one cell is active at any given time, while the others are quiescent. See Figure 4(b). We found this to be also the case in networks with larger number of cells. In particular, we conducted simulations with up to 20 cells.

In the following two sections, we investigate the existence of cycling behavior in coupled systems of near-identical cells. By “near-identical” we mean cells whose internal dynamics is governed by identical equations but with possibly different parameter values. We will consider two types of cells. First, cells whose internal dynamics is modeled by continuous-time systems of differential equations, and then cells modeled by discrete-time maps. In both cases, we will show that, under certain conditions, a network of near-identical cells can also produce cycling behavior. These type of cycles, however, are “richer” in the sense that a single trajectory can now connect different type of solutions such as steady-states (fixed points), periodic solutions (periodic orbits), and chaotic attractors as well.

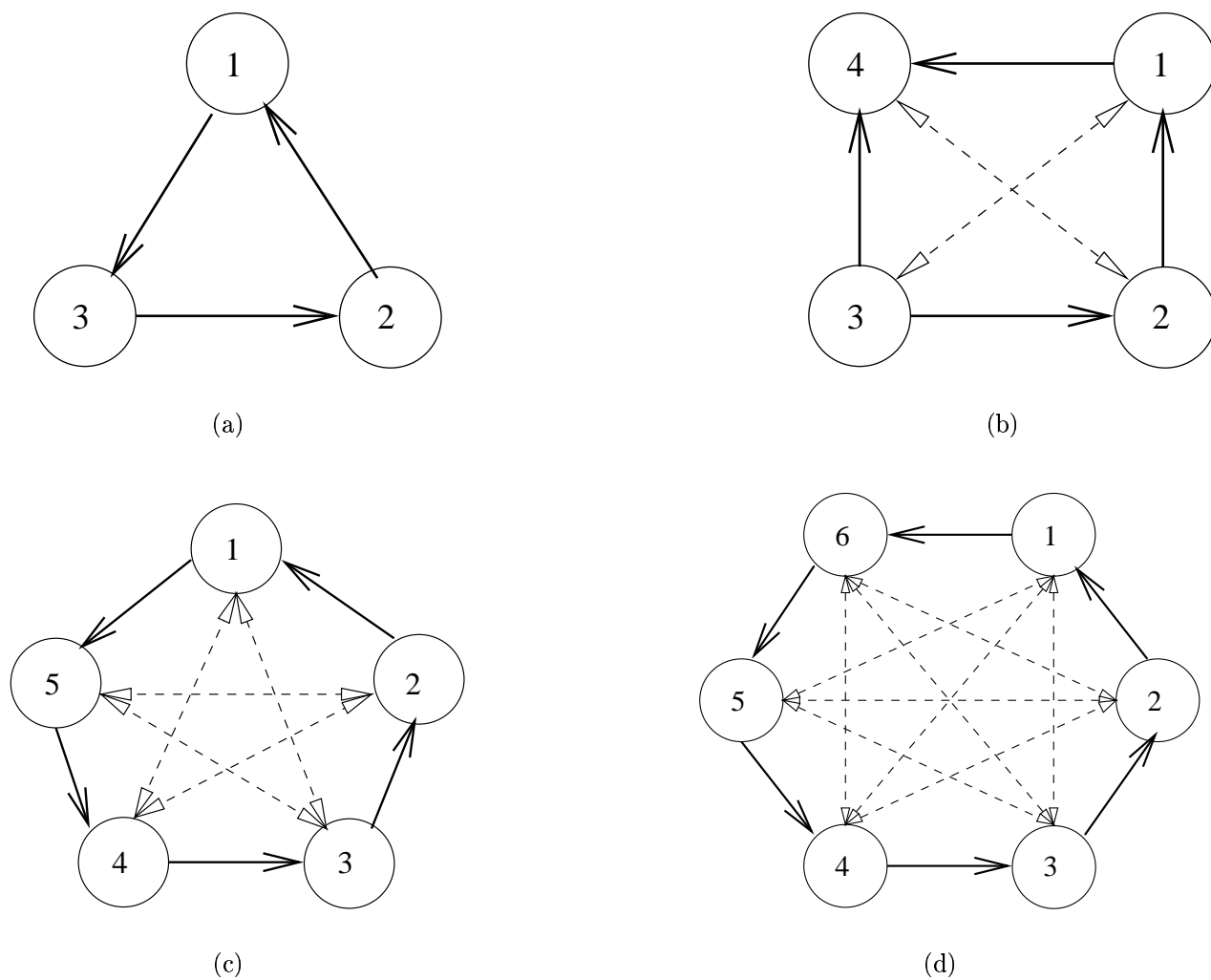


Figure 3: Interconnection scheme that supports cycling chaos. Nearest neighbors are connected in a directed ring fashion, while all remaining cells are connected in an all-to-all fashion. All couplings are identical.

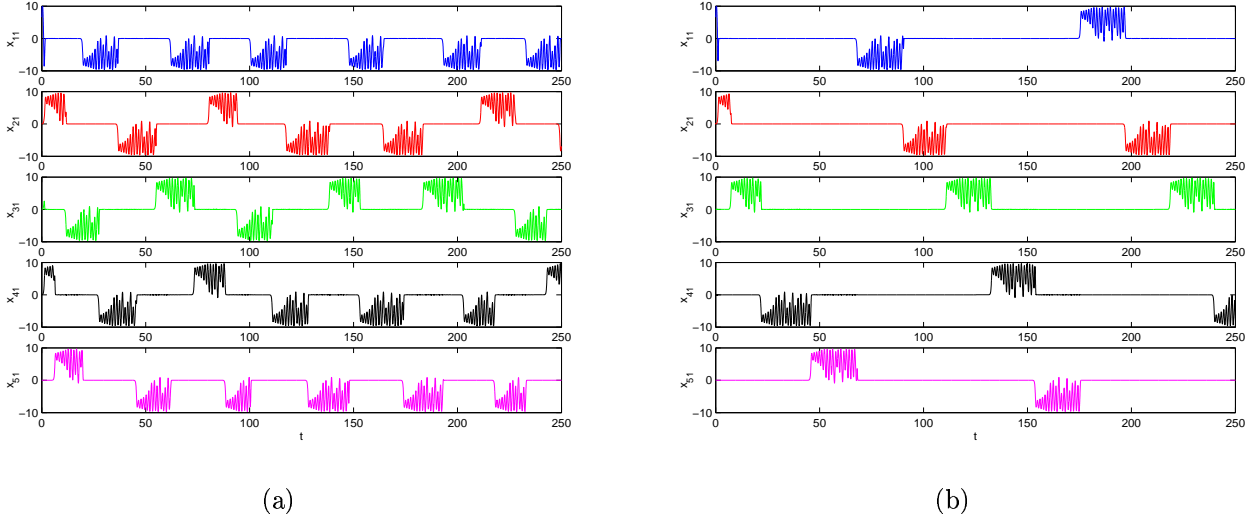


Figure 4: Simulations of a network of five identical cells, where the internal dynamics of each cell is modeled by a modified version of Chua’s circuit equations. (a) When only nearest neighbors are connected in a directed ring fashion, two or more cells can be simultaneously active. (b) When non-nearest neighbors are connected in an all-to-all fashion, however, only one cell is active at any given time, while the others are quiescent.

3 Continuous-Time Near-Identical Cell Systems

3.1 Modeling Equations

We now consider cells whose internal dynamics is modeled by continuous-time systems of differential equations. That is, for $1 \leq i \leq N$

$$\frac{dX_i}{dt} = f(X_i, \lambda_i), \quad (5)$$

where $X_i = (x_{i1}, \dots, x_{ik}) \in \mathbf{R}^k$ denotes the state variables of cell i , N is the total number of cells, $\lambda_i = (\lambda_{i1}, \dots, \lambda_{ip})$ is a vector of parameter values that controls the internal dynamics of cell i , f is smooth and independent of i since the cells are near-identical. Observe that although each cell is governed by the same equation, the dependence of the parameter vector λ on the cell number i allows for each uncoupled cell to exhibit its own unique internal dynamics.

A network is now a collection of near-identical interconnected cells, which can be modeled by a system of coupled differential equations of the form

$$\frac{dX_i}{dt} = f(X_i, \lambda_i) + \sum_{j \rightarrow i} \alpha_{ij} h(X_i, X_j), \quad (6)$$

where h is the coupling function between two cells, the summation is taken over those cells j that are coupled to cell i , and α_{ij} is a matrix of coupling strengths.

3.2 Simulations

Numerical simulations with networks similar to those depicted in Figure 3 were conducted. In particular, we conducted simulations of a network with $N = 4$ cells, with internal dynamics modeled by Chua's equations (3), and then of a network with $N = 6$ cells, where the internal dynamics of cell i is modeled by Rossler's equations

$$\begin{aligned}\dot{x}_{i1} &= -x_{i2} - x_{i3} \\ \dot{x}_{i2} &= x_{i1} + ax_{i2} \\ \dot{x}_{i3} &= b + (x_{i1} - c_i)x_{i3}.\end{aligned}\tag{7}$$

In both cases, wreath product coupling $h(X_i, X_j) = \|X_j\|X_i$ of identical strength given by $\alpha_{ij} = -\gamma$, where $\gamma > 0$, was used. Figure 5(a) shows the results of simulations of the network with four Chua's cell equations. Observe that, at any given time, only one cell is active while the others are quiescent. Parameter values that are identical in all cells are: $\alpha = 15$, $m_0 = -0.230769$, $m_1 = 0.0123077$, and $\gamma = 0.2$. We then varied β from cell 1 through 4 as follows, $\beta_1 = 30$, $\beta_2 = 53$, $\beta_3 = 33.136$, and $\beta_4 = 35$, respectively. This particular choice of β values yields internal dynamics in cells 1-4 that consists of a double scroll chaotic attractor, period-1 solution, another chaotic attractor, and a period-2 solution, respectively. The overall behavior of the interconnected network then yields a cycling trajectory that connects all of these solutions. Figure 5(b) shows similar results with cells whose internal dynamics is described by Rossler's equations (7). In this latter case, identical parameter values are: $a = 0.1$, $b = 0.1$, and $\gamma = 0.2$. Distinct values for parameter c from cell 1 through 6 are: $c_1 = 9$, $c_2 = 13$, $c_3 = 18$, $c_4 = 4$, $c_5 = 12$, $c_6 = 12.8$. This last choice of parameters causes (7) to yield three chaotic attractors, and periodic solutions of period 1, 3, and 6, respectively. These solutions become connected by the overall cycling behavior of the network.

4 Discrete-Time Near-Identical Cell Systems

4.1 Network Equations

We consider again systems with N near-identical cells, in the same sense as before, except that now the internal dynamics of each cell is governed by a k -dimensional difference equation of the form

$$X_{i_{n+1}} = f(X_{i_n}, \lambda_i)\tag{8}$$

where $X_i = (x_{i_1}, \dots, x_{i_k}) \in \mathbf{R}^k$ denotes the state variable of cell i and $\lambda_i = (\lambda_{i_1}, \dots, \lambda_{i_p})$ is a vector of parameters. A network of N cells is now modeled by a system of coupled difference

equations of the form

$$X_{i_{n+1}} = f(X_{i_n}, \lambda_i) + \sum_{j \rightarrow i} \alpha_{ij} h(X_{i_n}, X_{j_n}), \quad (9)$$

where h is the coupling function between those cells j that are coupled to cell i , $1 \leq i \leq N$, and α_{ij} represents the strength of the coupling. Observe that f is independent of i because the cells are assumed to be identical. Similarly, h is also independent of both i and j due to identical coupling.

4.2 Cells with Local \mathbf{Z}_2 Symmetry

As a representative example, we consider a network of four cells interconnected as is shown in Figure 3(b). The internal dynamics of each individual cell is governed by the \mathbf{Z}_2 -symmetric cubic map

$$f(x, \lambda) = \lambda x - x^3, \quad \lambda > 0, \quad (10)$$

where $\mathbf{Z}_2 = \{1, -1\}$. The bifurcation diagram of Figure 6 depicts the long-term dynamics of orbits for values of λ in the range $0 \leq \lambda \leq 3$. A wide range of complex behavior can be observed in this diagram, including period-doubling cascades and chaotic attractors. In fact, the bifurcations in (10) are reminiscent of those found in the *logistic* map [22], except that now local \mathbf{Z}_2 -symmetry forces two nontrivial fixed points (one with $x > 0$ and one with $x < 0$) to bifurcate from the trivial solution $x = 0$ at $\lambda = 1$. Each fixed point, in turn, undergoes a period-doubling cascade leading to a pair of chaotic attractors. Local \mathbf{Z}_2 symmetry again forces the cascades to occur at the same parameter values for each fixed point [4]. For $\lambda < \lambda_c = 3\sqrt{3}/2$, the attractors are confined to opposite sides of the $x = 0$ axis and each attractor has its own basin of attraction. At $\lambda = \lambda_c$, the basins of attraction collide and the two attractors merge into a single one. See Rogers and Whitley [29] for a more comprehensive analysis of a similar map $f(x, a) = ax^3 + (1 - a)x$, $0 \leq a \leq 4$.

To form the interconnected network equations (9), we consider a wreath product coupling function of the form

$$h(x_i, x_j) = |x_j|^m x_i, \quad (11)$$

where $0 < m < 1$. We will assume identical coupling strength given by $\alpha_{ij} = -\gamma$, where $\gamma > 0$. Observe that, as expected, h is equivariant under the \mathbf{Z}_2 action. The four-cells network, which possesses local \mathbf{Z}_2 -symmetry and global \mathbf{Z}_3 -symmetry, then takes the form

$$\begin{aligned} x_{1_{n+1}} &= \lambda_1 x_{1_n} - x_{1_n}^3 - \gamma(|x_{2_n}|^m + |x_{3_n}|^m)x_{1_n} \\ x_{2_{n+1}} &= \lambda_2 x_{2_n} - x_{2_n}^3 - \gamma(|x_{3_n}|^m + |x_{4_n}|^m)x_{2_n} \\ x_{3_{n+1}} &= \lambda_3 x_{3_n} - x_{3_n}^3 - \gamma(|x_{4_n}|^m + |x_{1_n}|^m)x_{3_n} \\ x_{4_{n+1}} &= \lambda_4 x_{4_n} - x_{4_n}^3 - \gamma(|x_{1_n}|^m + |x_{2_n}|^m)x_{4_n}. \end{aligned} \quad (12)$$

The value of the coupling strength γ and the parameter m are critical for the creation of cycling behavior because they control the global dynamics away from the internal dynamics of an individual cell. More specifically, the fact that $0 < m < 1$ prevents the global dynamics

from escaping to infinity and controls the rate at which the excursions from the dynamics of one cell to the next one occur. As m decreases, a typical orbit near a cycle spends longer time lingering around the dynamics of an active cell before it makes an excursion to the dynamics of the next cell.

Numerical simulations of (12) with various combinations of parameter values were conducted. For $1 \leq \lambda_i \leq \lambda_c$, and appropriate values of coupling strength γ and parameter m , the simulations yield trajectories that cycle around the orbits generated by the internal dynamics of each individual cell i , according to the actual value of λ_i . For instance, Figure 7 depicts the results of one particular simulation with initial conditions $(x_{1_0}, x_{2_0}, x_{3_0}, x_{4_0}) = (-0.01, 0.03, 0.02, 0.5)$, and parameter values $\gamma = -3.05$, $m = 1/3$, $\lambda_1 = 2.602$, $\lambda_2 = 2.2$, $\lambda_3 = 2.44$, and $\lambda_4 = 2.26$. According to Figure 6, $\lambda_1 = 2.602$ is near the critical point λ_c , where two symmetrically related chaotic attractors collide. In this region, the internal dynamics of cell 1 produces an intermittent chaotic orbit that switches between the remnants of the two attractors [19]. Similarly, cell 2 generates a period-2 orbit, and cell 3 yields a chaotic attractor that fills parts of the interval $[0, 2]$. Local \mathbf{Z}_2 symmetry in the internal dynamics of cell 3 also forces the existence of a conjugate chaotic attractor within $[-2, 0]$. Finally, the long-term dynamics of cell 4 is a period-4 orbit. Observe that all these solutions are visited by one single trajectory, and that, at any given time, only one cell is active in its own internal dynamics, while the others are quiescent.

We have also found that if λ_i lies within the interval $1 \leq \lambda \leq \lambda_c$, and depending on the coupling strength, then when cell i becomes active it can exhibit one of two types of behavior. Either the active cell always selects the same of two conjugate orbits ($x > 0$ or $x < 0$), or it can switch intermittently between conjugate orbits. In the former case, the actual sign of the orbit that is selected depends on the initial conditions of the active cell. In Figure 7, cells 1-3 switch intermittently when they become active.

5 Conclusions

We have presented a network connectivity that supports the existence of cycling behavior in coupled systems of identical and near-identical cells with more than three cells, and where the cell dynamics is modeled by either continuous- or discrete-time systems of differential or difference equations. In particular, we have shown that cycling behavior in coupled systems of near-identical cells, as is the case with identical cells, can also be a global phenomenon that persists independently of the local dynamics of individual cells. Even when distinct parameter values force each cell to generate different long-term behavior. This type of cycles offer a wider range of behavior in the sense that a single trajectory can now visit multiple types of solutions or orbits, including chaotic attractors.

Acknowledgments

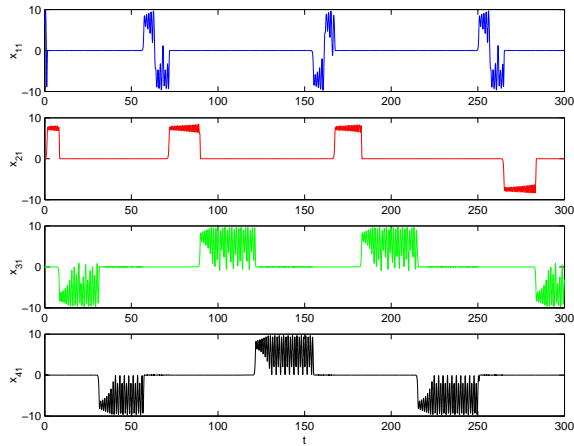
I would like to thank Marty Golubitsky and Ian Melbourne for many stimulating conversations.

References

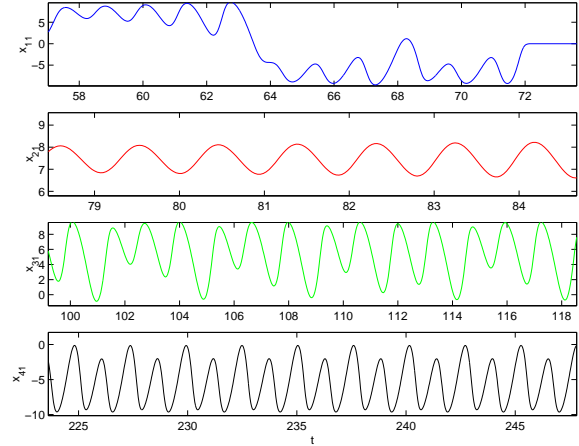
- [1] Armbruster, D., Guckenheimer, J. & Holmes, P. [1988] “Heteroclinic cycles and modulated traveling waves in systems with $O(2)$ symmetry,” *Physica D* **29**, 257-282.
- [2] Aronson, D.G., Golubitsky, M. & Krupa, M. [1991] “Coupled arrays of Josephson junctions and bifurcation of maps with S_N symmetry,” *Nonlinearity* **4**, 861-902.
- [3] Buono, P.L., Golubitsky, M. & A. Palacios. [2000] “Heteroclinic cycles in rings of coupled cells,” *Physica D* **143**, 74-108.
- [4] Chossat, P. & Golubitsky, M. [1988] “Iterates of maps with symmetry,” *SIAM J. Math. Anal.* **19**(6), 1259-1270.
- [5] Dellnitz, M., Field, M., Golubitsky, M., Ma, J. & Hohmann, A. [1995] “Cycling chaos,” *Int. J. Bifurc. Chaos* **5**(4), 1243-1247.
- [6] Dionne, B. Golubitsky, M., & Stewart, I. [1996] “Coupled cells with internal symmetry. Part I: wreath products,” *Nonlinearity* **9**, 559-574.
- [7] Dionne, B. Golubitsky, M., & Stewart, I. [1996] “Coupled cells with internal symmetry. Part II: direct products,” *Nonlinearity* **9**, 575-599.
- [8] Doedel, E.J., Aronson, D.G. & Othmer, H.G. [1988] “The dynamics of coupled current-biased Josephson junctions: Part I,” *IEEE Trans. Circuits and Systems* **35**(7), 810-817.
- [9] Field, M.J. [1980] “Equivariant dynamical systems,” *Trans. Amer. Math. Soc.* **259**(1), 185-205.
- [10] Field, M.J. [1996] *Lectures on Bifurcations, Dynamics and Symmetry*. Pitman Res. Notes **356**, Addison-Wesley Longman Ltd., Harlow.
- [11] Guckenheimer, J. & Holmes, P. [1988] “Structurally stable heteroclinic cycles,” *Math. Proc. Camb. Phil. Soc.* **103**, 189-192.
- [12] Golubitsky, M. & Stewart, I. [1999] “Symmetry and pattern formation in coupled cell networks. In: *Pattern Formation in Continuous and Coupled Systems*, (M. Golubitsky, D. Luss and S.H. Strogatz, eds.) IMA Volumes in Mathematics and its Applications **115**, Springer, New York, 65-82.

- [13] Halow, J.S., Boyle, E.J., Daw, C.S. & Finney, C.E.A. [1998] "PC-based, near real-time, 3-dimensional simulation of fluidized beds," *Fluidization IX Durango, Colorado*.
- [14] Hadley, P., Beasley, M.R. & Wiesenfeld, K. [1988] "Phase locking of Josephson-junction series arrays," *Phys. Rev. B* **38**, 8712-8719.
- [15] Kopell, N. & Ermentrout, G.B. [1988] "Coupled oscillators and the design of central pattern generators," *Math. Biosci.* **89**, 14-23.
- [16] Kopell, N. & Ermentrout, G.B. [1990] "Phase transitions and other phenomena in chains of oscillators," *SIAM J. Appl. Math.* **50**, 1014-1052.
- [17] Krupa, M. [1997] "Robust heteroclinic cycles," *J. Nonlin. Sci.* **7** (2), 129-176.
- [18] Krupa, M. & Melbourne, I. [1995] "Asymptotic stability of heteroclinic cycles in systems with symmetry," *Ergod. Th. & Dynam. Sys.* **15**, 121-147.
- [19] Manneville, P. [1990] *Dissipative Structures and Weak Turbulence*. (Academic Press, Inc.)
- [20] May, R.M. [1974] "Biological populations with no overlapping generations: stable points, stable cycles, and chaos. *Science* **186**, 645-647.
- [21] May, R.M. [1975] "Biological population obeying difference equations: stable points, stable cycles, and chaos. *J. Theor. Biol.* **51**, 511-524.
- [22] May, R.M. [1976] "Simple mathematical models with very complicated dynamics. *Nature* **261**, 459-467.
- [23] Melbourne, I., Chossat, P. & Golubitsky, M. [1989] "Heteroclinic cycles involving periodic solutions in mode interactions with $O(2)$ symmetry," *Proc. Roy. Soc. of Edinburgh* **113A**, 315-345.
- [24] Palacios, A. [2002] "Cycling chaos in one-dimensional coupled iterated maps," To appear in *Int. J. Bifurc. Chaos* **12**(8).
- [25] Palacios, A. [2002] "Heteroclinic cycles in coupled systems of difference equations," To appear in *Int. J. Diff. Eq. Appl.*
- [26] Pecora, L & Carroll, T.L. [1990] "Synchronization in chaotic systems," *Phys. Rev. Lett.* **64**, 821-824.
- [27] Rappel, W. [1994] "Dynamics of a globally coupled laser model," *Phys. Rev. E.* **49**, 2750-2755.
- [28] Rand, R.H., Cohen, A.H. & Holmes, P.J. [1988] Systems of coupled oscillators as models of central pattern generators. In: *Neural Control of Rhythmic Movements in Vertebrates* (A.H. Cohen, S. Rossignol and S. Grillner, eds), Wiley, New York, 333-367.

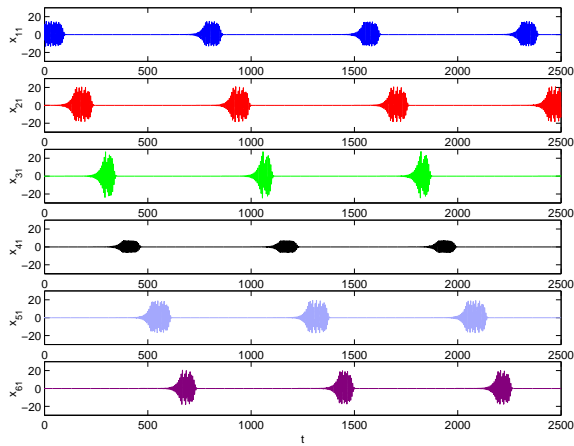
- [29] Rogers, T. & Whitley, D.C. [1983] “Chaos in the cubic mapping,” *Math. Modelling* **4**, 9-25.
- [30] Toner, J. & Yuhai, T. [1998] “Flocks, herds and schools: A quantitative theory of flocking. *Phys. Rev. E* **58**, 4828-4858.
- [31] Wiesenfeld, K., Bracikowski, C., James, G. & Rajarshi, R. [1990] “Observation of antiphase states in a multimode laser,” *Phys. Rev. Lett.* **65**(14), 1749-1752.
- [32] Wu, CW & Chua, L.O. [1994] “A unified framework for synchronization and control of dynamical systems,” *Int. J. Bifurc. Chaos* **4**(4), 979-998.



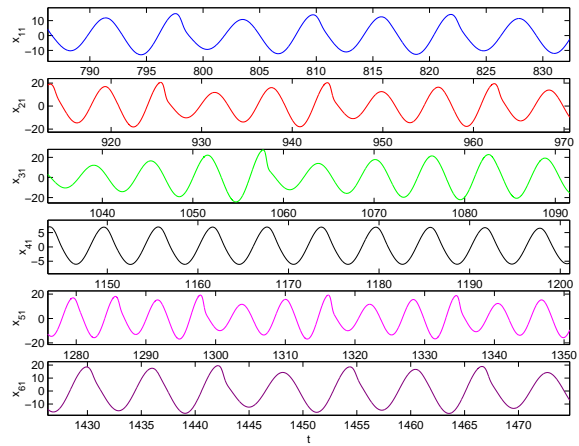
(a) Cycling behavior in Chua's circuit.



(b) Magnification of active phase in (a)



(c) Cycling behavior in Rossler's equations



(d) Magnification of active phase in (c)

Figure 5: Cycling behavior in networks of coupled near-identical cells. (a) Cycling trajectory connecting chaotic set, period-1 solution, chaotic set, and period-2 solution of Chua's circuit. (b) Cycling trajectory connecting chaotic sets, period-1, period-3, and period-6 solutions of Rossler's equations.

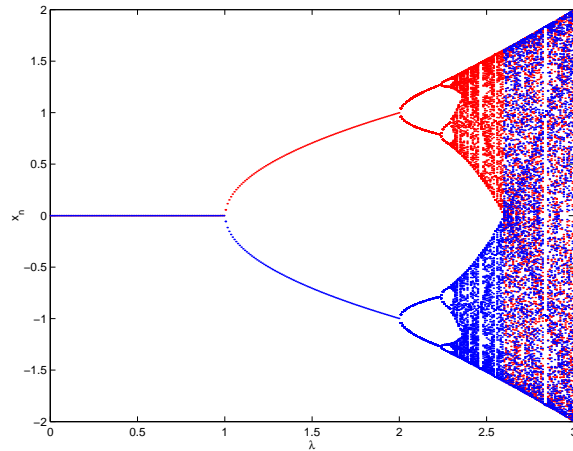
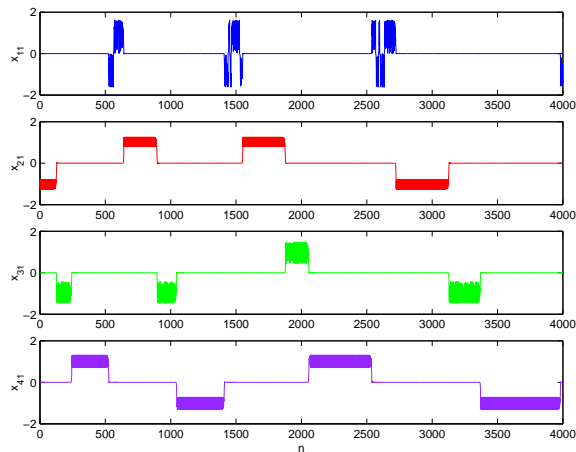
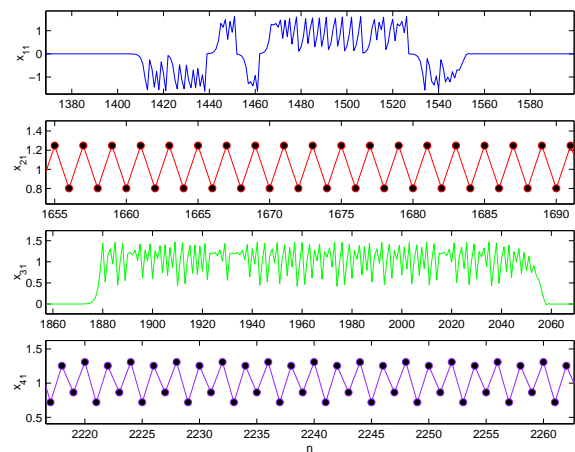


Figure 6: Bifurcation diagram for a cell with internal dynamics $f(x, \lambda) = \lambda x - x^3$.



(a) Cycling behavior in cubic map.



(b) Magnification of active phase in (a)

Figure 7: Cycling behavior in a network of four near-identical discrete cells. Cycling trajectory connects (in order) an intermittent chaotic attractor with a period-2 orbit with another chaotic set with a period-4 orbit and back to the intermittent chaotic attractor.