Approximating with the Liouville-Green

D. W. Verzi[1]

Running Head: Liouville-Green Asymptotics

[1]Department of Mathematics, San Diego State University-Imperial Valley Campus

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D. W. Verzi, Department of Mathematics and Statistics,
San Diego State University-Imperial Valley Campus,
720 Heber Avenue, Calexico CA 92231
Phone: (760)768-5531
Fax: (760)768-5631
Email: verzi@math.sdsu.edu

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Abstract

Among the methods used to approximate solutions for second-order differential equations, one of the oldest and least understood is the Liouville-Green. Liouville and Green first introduced this technique in 1837, but their work was relatively unnoticed until Jeffreys and Kramer rediscovered it in the early 1920s. Much has been written in recent years to add to the actual theory, with a major contribution in the area of error analysis in the 1960s. Since then, the Liouville-Green has been applied with greater confidence to equations that cannot be solved in elementary terms. Rather than using the method to approximate solutions for a more difficult equation, this paper looks specifically at the method, by applying it to solvable equations, so that results may be compared graphically.

Euler's equation and a special case for Bessel's equation are reduced to forms necessary for approximation, and singularities are identified. The coefficient functions are partitioned several times to produce both acceptable and unacceptable approximations in the neighborhood of singularities, as predicted by the error control function. An effective partition allows the Liouville-Green to return an asymptotic solution that is equivalent to the exact solution for Euler's equation when parameter values imply exponential or oscillatory solutions. For Bessel's equation, partitions of the coefficient function produce solutions that are applicable in distinct regions of the domain, and a final approximation that is uniformly acceptable for all positive input.

Key Words: Liouville-Green, JWKB, asymptotics, error analysis, Bessel's equation

1 Introduction

In some circles of mathematics, asymptotic methods have a "bad" reputation. They are introduced in upper level applied mathematics courses, where rigorous exposition and domains of application may not be fully appreciated by the engineers and scientists who will ultimately apply them [4]. However, for practical purposes, many partial and ordinary differential equations are intractable to closed-form solutions and contain terms and parameters that are themselves the result of asymptotic or limiting approximations. With increased availability of high-speed computers, numerical computations have become the norm, rather than the exception, in resolving a large percentage of these equations. To those who work in the field of asymptotics, "this is like passing from [decadence to barbarism], without experiencing civilization" [in between] [4]. Moreover, numerical methods experience difficulty in singular areas, and their error increases with dimension. On the other hand, asymptotic methods are the result of rigorous analysis for simplifying assumptions that are well-defined during exposition. With any method of mathematics, if asymptotics are applied outside their domain of definition, unreasonable solutions may be mistaken for those that are reasonable. When properly applied, asymptotic solutions are elegant in their own right, and they deliver computer-generated data in a much shorter period of time than direct numerical resolution of the differential equation [1].

This paper takes a close look at the Lioville-Green approximation (LG) method for second-order differention equations: What it is, how it works, its reliability, and its ability to "imitate" solutions requiring analytic or numeric approximation for practical applications. The importance of asymptotics to the theory of differential equations was recognized in the early 1800s, but their application to problems in quantum mechanics, viscous flows, elasticity, electromagnetic theory, electronics, and astro-physics grew out of a need for numerical analysis of computer output in the last half of the 20th century. Asymptotics help us understand the nature of these equations and point the way to the structure of their solutions [2, 9].

Liouville and Green first introduced this technique in 1837, but their work was relatively unnoticed until Jeffreys and Kramer rediscovered it in the early 1920s [7] [8]. In 1926, Brillouin, Wentzel and Kramer applied the method to Schrodinger's wave equation and, in their honor, the method is often referred to as the JWKB. It has also been called the "phase integral" technique [5]. The LG approximates solutions for equations of the form $w"(x) = h(x)w(x)$, where x is real or complex and $h(x)$ is at least twice differentiable. A particular strength of the LG is its ability to approximate a solution in the neighborhood of a singularity, where numerical methods frequently experience instabilities.

The method was widely compared to numerical results prior to 1961, but still suspect with regard to error until Olver proved that convergence of the error control function establishes a reliable measure for error (see [8] Ch 6.2). He then derived explicit error bounds that direct the optimal partition of the coefficient function [8]. Since then, the LG has been applied with greater confidence to equations which cannot be solved in elementary terms. More recently, Dunster, Lutz and Schäfke formulated a convergent LG expansion [3], and Geronimo and Smith developed a discrete analog of LG expansions for Ricatti difference equations [6]. Current work considers necessary and sufficient conditions for summability of LG solutions. However, those applying LG in the fields of science and engineering still believe the method to be vague, particularly with respect to transformation and partitioning.

Rather than using the LG to approximate the solution for a more difficult equation, this paper looks specifically at the method by applying it to Euler's equation and a solvable case for Bessel's equation, so that results may be compared to exact solutions. The paper is organized as follows: The Formulation section transorms the general second-order differential equation to a form for LG approximation. In the sections concerning Euler and Bessel equations, different partitions of the coefficient functions are analyzed and results are compared to exact solutions. The last section is Discussion.

2 Formulation

The LG method approximates a solution for second order differential equations of the form

$$\frac{d^2w}{dx^2} = h(x)w, \tag{1}$$

where x is a real or complex variable and $h(x)$ is a prescribed function. Any homogeneous, second-order, linear differential equation may be stated in this form by appropriately redefining the dependent variable. If $h(x)$ is real, positive, and slowly varying, the general solution for w is expected to be a linear combination of two exponentials. If $h(x)$ is negative, the solution is expected to be oscillatory in nature.

To apply the method for $h(x)$ positive, the coefficient function is partitioned as

$$h(x) = f(x) + g(x), \tag{2}$$

where $f(x)$ is positive, real, and twice continuously differentiable, and $g(x)$ is a continuous real or complex function [7] Ch 6.2, [8] Ch 6.1. Once an equation has been so defined, the LG approximates solutions as

$$w_1(x) = f^{-1/4}(x)A\exp\Big[\int f^{1/2}(x)\,dx\Big](1+\epsilon_1(x)) \tag{3}$$

$$w_2(x) = f^{-1/4}(x)B\exp\Big[-\int f^{1/2}(x)\,dx\Big](1+\epsilon_2(x)), \tag{4}$$

where A and B are arbitrary constants, dependent upon initial or boundary conditions. The error in this scheme is bounded by

$$|\epsilon_j(x)| \leq \exp[\frac{1}{2}V_j(x)] - 1, \tag{5}$$

where

$$V_j(x) = \int_{a_j}^{x}\left|\frac{1}{f^{1/4}(t)}\frac{d^2}{dt^2}\left[\frac{1}{f^{1/4}(t)}\right] - \frac{g(t)}{f^{1/2}(t)}\right|dt, \tag{6}$$

for $j = 1, 2, ...$ assigning a counting number to the singularities, and a_j representing the particular point. With appropriate choices for f and g, the LG provides reasonable asymptotic representations, even in the region of a singularity.

To utilize the method for any given linear second-order differential equation, one should first identify points of singularity, and then convert the equation to the form of Eq. (1). The general linear second-order differential equation may be expressed as

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{7}$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials with no common factors. Points of singularity occur when $P(x) = 0$. If the singularity is regular, the solution behaves algebraically nearby; solutions near irregular singularities behave exponentially. For details on classification of singularities, see [8].

2.1 Redefining the variables

A monic second-order linear differential equation takes the form

$$y"(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{8}$$

To recast the above into a form for LG approximation, one must eliminate the coefficient function for y'. For exponential or oscillatory solutions, it is appropriate to seek a product solution, $y(x) = \phi(x)w(x)$. If

$$\phi(x) = \exp\Big[\frac{1}{2}\int p(x)\,dx\Big], \tag{9}$$

then

$$w" = \Big(\frac{1}{4}p^2 + \frac{1}{2}p' - q\Big)w, \tag{10}$$

takes the form of a second-order differential equation whose solution may be approximated using LG asymptotics.

3 Euler's Equation

Consider Euler's equation

$$x^2\frac{d^2z}{dx^2} + \alpha x\frac{dz}{dx} + \beta z = 0, \tag{11}$$

where α and β are constants.

Euler's equation has exact solutions of the form

$$z_r = c_1 x^{r_1} + c_2 x^{r_2},$$

where

$$r_i = \frac{1-\alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2} \tag{12}$$

The nature of these solutions will be considered later, and compared to the nature of LG approximations, for different values of the parameters α and β.

Since $P(x)$ from Eq. (7) equals x^2, it is clear that $x = 0$ is a singular point. To see that $x = \infty$ is also singular, consider the transformation $x = 1/t$. Then Eq. (11) may be written in terms of t as

$$t^2 \frac{d^2 z}{dx^2} + t(2-\alpha)\frac{dz}{dx} + \beta z = 0. \tag{13}$$

The behavior of (13) at $t = 0$ determines the behavior of (11) at $x = \infty$. Since $P(t = 0) = 0$, then $x = \infty$ is also a point of singularity for (11). The solution behaves algebraically near both singularities, so that $x = 0$ and $x = \infty$ are regular singular points [8].

For Euler's equation, the coefficient functions in (8) are

$$p(x) = \frac{\alpha}{x}$$

$$q(x) = \frac{\beta}{x^2},$$

so that Eq. (10) permits the following transformation to a form for the LG method

$$w" = \frac{\lambda}{x^2} w,$$

where

$$\lambda = \frac{1}{4}\big(\alpha^2 - 2\alpha - 4\beta\big).$$

In addition, the product function is obtained from Eq. (9) as,

$$\phi(x) = \exp\big(\frac{-\alpha \log|x|}{2}\big)$$

so that for $x > 0$, the LG approximation for Euler's equation may be obtained from the product

$$z(x) = x^{-\alpha/2} w(x). \tag{14}$$

3.1 Results

In this section, Euler's equation is approximated near the singularity at $x = 0$ for two distinct partitions of $h(x)$ to compare results to the exact solution. The initial partition produces an unacceptable error. Then analysis of the error control function (6) leads to a second partition that uniformly approximates the solution for $x > 0$.

Note that in Eq. (2)-(6), the role of $g(x)$, in general, is to reduce the functional value for $f(x)$, but $g(x)$ becomes an "active" participant in the error control function. To approximate the solution near zero, it seems natural to begin with a partition $h(x) = f_1(x) + g_1(x)$ to minimize $g_1(x)$ for smaller input values. Let

$$f_1(x) = \frac{\lambda}{x^2}$$

$$g_1(x) = 0.$$

Then

$$f_1^{-1/4} = \lambda^{-1/4} x^{1/2},$$

and

$$\xi = \int \sqrt{f_1} = \lambda^{1/2} \log(x).$$

The factors of λ may be absorbed into the arbitrary constants in the LG approximation (3) to arrive at the asymptotic solution for $x > 0$

$$w_1(x) = x^{1/2} \big[A \exp(\xi) + B \exp(-\xi) \big],$$

neglecting error terms. Identifying inverse functions and utilizing Eq. (14) for the product function, the LG approximation becomes

$$z_1(x) = A x^{(1-\alpha+2\sqrt{\lambda})/2} + B x^{(1-\alpha-2\sqrt{\lambda})/2}. \tag{15}$$

However, this approximation does not compare well to the exact solution. Figure 1a graphs the exact solution, and Eq. (15) for $\alpha = -2$ and $\beta = -1$, with values for A and B determined by the functional value and slope at $x = 1$. The approximation fits the exact solution in a small neighborhood of initial data, but quickly diverges in both directions.

To see why (15) does not give an adequate approximation, consider the error control function for the above partition. For $f_1(x)$ and $g_1(x)$, Eq. (6) evaluates to

$$V(x) = \int_0^x \left| -\frac{1}{4}\lambda^{-1/4}x^{1/2}\lambda^{-1/4}x^{-3/2} \right| dx = \frac{1}{4}\lambda^{-1/2}\log(x). \tag{16}$$

Since $\log(x)$ is not bounded near zero or ∞, $V(x)$ does not converge, and the partition fails to produce an acceptable approximation. One normally matches coefficients in the limit as x approaches a singularity. However, there are no coefficients A, B that permit Eq. (15) to approximate the exact solution as $x \to 0$ or ∞.

For a second attempt, consider a partition that is directed by the error control function. Let

$$f_2(x) = \frac{\lambda - \eta}{x^2} \tag{17}$$

$$g_2(x) = \frac{\eta}{x^2}, \tag{18}$$

where η is a parameter to be identified for convergence. Then

$$V_j(x) = \int_0^x \left| \frac{-1+4\eta}{4(\lambda-\eta)^{1/2}t} \right| dt = \left| \frac{-1+4\eta}{4(\lambda-\eta)^{1/2}} \right| \log(x), \tag{19}$$

for $x > 0$. Observe that $V(x)$ is bounded if and only if $1 + 4\eta = 0$. If $\eta = -1/4$, then

$$f_2(x) = \frac{\lambda + \frac{1}{4}}{x^2}.$$

Set

$$\tilde{\lambda} = (\lambda + \frac{1}{4})^2,$$

so that

$$f_2^{-1/4} = \tilde{\lambda}^{-1/8}x^{1/2},$$

and

$$\xi = \int \sqrt{f_2} = \tilde{\lambda}^{1/4} \log(x).$$

When these functions are inserted into the LG,

$$w_2(x) = Ax^{(1+2\tilde{\lambda}^{1/4})/2} + Bx^{(1-2\tilde{\lambda}^{1/4})/2},$$

and Euler's equation is approximated as

$$z_2(x) = Ax^{(1-\alpha+2\tilde{\lambda}^{1/4})/2} + Bx^{(1-\alpha-2\tilde{\lambda}^{1/4})/2}. \tag{20}$$

Since the error bound in (19) is exactly zero for $\eta = -1/4$, the asymptotic solution is not just a good approximation near the singularity at zero, but equivalent to the exact solution for all $x > 0$.

3.2 Nature of Solutions

With such positive results as those above, one wonders if Euler's equation may always be well-approximated, regardless of parameter values. The exact solution, Eq. (12), was obtained by seeking a power solution of the form

$$z = x^r$$

for $r > 0$. Substitution yields an equation in x and r,

$$r(r-1)x^r + \alpha r x^r + \beta x^r = 0.$$

Suppose the parameters α and β in (11) are real and $x > 0$. Then $x^r \neq 0$ and the roots of the above indicial equation,

$$r^2 + (\alpha - 1)r + \beta = 0, \tag{21}$$

describe exponential powers for the general solution to Euler's equation.

The discriminant for the above quadratic determines the nature of Euler's solutions. If $(\alpha-1)^2 - 4\beta > 0$, the general solution is a sum of two linearly independent exponentials, with one dominating as $x \to 0$, and the other dominating as $x \to \infty$. The LG approximation for this case will also be exponential, a linear combination of two real powers in x, as in Fig. 1. The partition for $h(x) = f(x) + g(x)$ was found to have an acceptable error when $g(x) = -1/(4x^2)$. Since $h(x)$ is assumed strictly positive, there is "room" in the partition for $f(x) > 0$. Recall from the Formulation section that for bounded error in the asymptotic solution, $f(x)$ must be real, positive and twice continuously differentiable for exponential approximations [8].

Suppose, instead, that the parameters in Euler's equation cause $(\alpha - 1)^2 - 4\beta < 0$. Then the general solution will be exponentials with complex powers, $r_i = \phi \underline{+} i\mu$, giving the general (real) solution

$$z = c_1 x^\phi \cos(\mu \log(x)) + c_2 x^\phi \sin(\mu \log(x)).$$

It has been shown above that the optimal LG occurs when $g(x) = -1/(4x^2)$. If $\lambda < -1/4$, the coefficient function $h(x) < -1/(4x^2) < 0$, and the partition requires $f(x) < 0$. In this situation, the general LG solution takes the form

$$w = \Big[|f|^{-1/4}(x)\Big]\Big[A \exp\big(i \int |f|^{1/2}(x)\big) + B \exp\big(-i \int |f|^{1/2}(x)\big)\Big].$$

The above exponentials are consistent with those in (20), which utilize the standard form of the LG. While Eq. (20) does not explicitly include imaginary components, it allows for complex numbers in the square and fourth roots of $f(x)$, and the asymptotic solution will likewise produce complex conjugates that may be converted to an oscillatory solution using Euler's identity.

Now suppose that the parameters are such that $(\alpha - 1)^2 - 4\beta = 0$. Then $r_1 = r_2 = (1-\alpha)/2$ is a double root for the indicial equation, yielding a power solution and a logarithmic solution that are linearly independent, so that the general solution takes the form

$$z_r = c_1 x^r + c_2 x^r \log(x).$$

However, this solution is neither exponential nor oscillatory, and does not match the form for LG asymptotics. In the case of repeated roots, Bessel functions are required for asymptotic approximation (see [8] Ch 12.) If the discriminant for Euler's equation is zero, then $\lambda = \eta$ and $f_2 \equiv 0$ in (18), so that the partition breaks down. In the general linear second-order differential equation, when $\lim_{x\to 0} x^2 h(x) = -1/4$, a suitable partition cannot be chosen for convergence of the error control function [8]. For Euler's equation, this occurs when the roots of (21) are equal.

4 Bessel's Equation

For a second example, consider the modified Bessel's equation

$$x^2 y(x)" + xy(x)' - (x^2 - \nu^2)y(x) = 0 \tag{22}$$

for $x > 0$, where ν is constant. The points of singularity occur at $x = 0$ and ∞, with $x = 0$ a regular singularity. However, the solution for Bessel's equation behaves exponentially as x get large, so that infinity is an irregular singular point [8].

As a special and solvable case, let $\nu = 3/2$. Then Eq. (9)-(10) from the Formulation section recasts Eq. (22) into a form for LG approximation as

$$w" = \left(1 + \frac{2}{x^2}\right)w, \tag{23}$$

where the solution for (22) takes the form

$$y(x) = \sqrt{x}w(x).$$

Since

$$h(x) = 1 + \frac{2}{x^2} > 0$$

in (23), solutions will be a linear combination of exponentials, with one dominating as $x \to 0$, and the other dominating as $x \to \infty$. Substitution confirms that the general solution for $w(x)$ takes the form

$$w = c_1\left(1 - \frac{1}{x}\right)\exp(x) + c_2\left(1 + \frac{1}{x}\right)\exp(-x).$$

For a natural choice of coefficients, let $c_1 = 1$ and $c_2 = 0$. Then

$$w_a = \left(1 - \frac{1}{x}\right) \exp(x)$$

represents one specific solution. For a second solution, let $c_1 = 0$, and $c_2 = 1$. Then

$$w_b = \left(1 + \frac{1}{x}\right) \exp(x). \tag{24}$$

However, near the singularity at $x = 0$, a Taylor expansion confirms

$$w_a(x) = \frac{-1}{x} + O(x) \tag{25}$$

and

$$w_b(x) = \frac{1}{x} + O(x), \tag{26}$$

so that both functions dominate as $x \to 0$, and they do not form a numerically satisfactory pair.

Instead, consider a solution where $c_1 = c_2 = 1/2$. Then

$$w_c(x) = \cosh(x) - \frac{\sinh(x)}{x} \tag{27}$$

represents one specific solution; for a second solution, keep (24). Near the singularity at $x = 0$, a Taylor expansion confirms

$$w_c(x) = \frac{2x^2}{3} + O(x^3) \tag{28}$$

so that w_c is recessive. Likewise, near the singularity at $x = \infty$,

$$w_b(x) = \exp(-x)\left(1 + O\left(\frac{1}{x}\right)\right) \tag{29}$$

and

$$w_c(x) = \exp(x)\left(\frac{1}{2} + O\left(\frac{1}{2x}\right)\right), \tag{30}$$

so that w_b is recessive and w_c dominates. Further, the Wronskian confirms that these solutions are linearly independent. Thus w_b and w_c form a numerically satisfactory pair of solutions for (22), and the general form may be written as

$$y_r(x) = \sqrt{x}w_r(x), \tag{31}$$

where

$$w_r(x) = \big(C_1 w_b(x) + C_2 w_c(x)\big). \tag{32}$$

4.1 Results

In this section, Bessel's equation is approximated for distinct partitions of $h(x)$. An initial partition produces an acceptable error near zero that diverges as $x \to \infty$. The second partition produces an acceptable error for large values of x that diverges as $x \to 0$. Finally, an analysis of the error control function yields a partition to approximate the solution for all $x > 0$.

To identify the optimal partition in a neighborhood of zero, let

$$f_1(x) = \frac{2}{x^2} + \frac{\eta}{x^2}$$

$$g_1(x) = 1 - \frac{\eta}{x^2},$$

where η is, once again, a parameter to be identified for convergence of the error control integral. With this partition, the integrand in (6) is asymptotic to

$$H(x) \approx \frac{x}{\sqrt{2+\eta}} + \frac{4\eta+1}{4x\sqrt{2+\eta}}.$$

For $\eta = 1/4$, Eq. (5)-(6) produce an error bound

$$\epsilon_1(x) \leq \exp(\frac{x^2}{6}) - 1,$$

convergent near zero, but increasing with x. Therefore, with

$$f_1(x) = \frac{9}{4x^2}$$

and

$$g_1(x) = 1 - \frac{1}{4x^2},$$

an acceptable LG approximation may be computed from (3) as

$$w_1(x) = Ax^2 + \frac{B}{x}.$$

The first term compares well to (27) near zero, when $A = 1/3$. However, the functions diverge as x gets large. Fig. 2a graphs the error bound (ϵ_1) against the relative error $|w_c - Ax^2|/w_c$, comparing the analytic and asymptotic solutions recessive near zero. While the above LG is asymptotic to the exact solution near zero, the approximation breaks down as x increases.

To find an acceptable approximation for large values of x, consider a partition that minimizes the magnitude of $g(x)$ as $x \to 0$. Let

$$f_2(x) = 1$$

and

$$g_2(x) = \frac{2}{x^2}.$$

This time, the "natural" choice for a partition is appropriate since (5)-(6) produce an error bound

$$\epsilon_2(x) \leq \exp(\frac{1}{x}) - 1,$$

convergent as $x \to \infty$, so that the LG approximation takes the form

$$w_2(x) = A\exp(x) + B\exp(-x).$$

The second term compares well to (24) for large values of x when $B = 1$, but the functions diverge as $x \to 0$. Fig. 2b graphs the error bound against the relative error $|(w_b - \exp(-x)|/w_b$, comparing analytic and asymptotic solutions recessive as $x \to \infty$. The above LG approximates the solution to (23) for large values of x, but the approximation breaks down as $x \to 0$.

The previous two partitions produce acceptable approximations for Bessel's equation near distinct singular points. However, a more powerful approach would be to produce a partition that delivers a LG uniformly acceptable for all $x > 0$. Let us, again, take an analytic approach to identify such a partition. Let

$$f_3(x) = 1 + \frac{2}{x^2} + \frac{\eta}{x^2} \tag{33}$$

and

$$g_3(x) = \frac{-\eta}{x^2},$$

where η is a parameter to be indentified so that the error is bounded near both singularities. This time (6) produces the error control function:

$$V_j(x) = \int_{a_j}^{x} |H(t)|\, dt. \tag{34}$$

where the integrand

$$H(x) = \frac{4\eta x^4 - 12x^2 + 10\eta x^2 + 8\eta^2 x^2 + 12\eta - 4 + 15\eta^2 + 4\eta^3}{4x(x^2 + 2 + \eta)^{5/2}}.$$

Fortunately, it is enough for $H(x)$ to be bounded near the points of singularity to assume that the error is bounded. For x near zero, a Taylor expansion yields

$$H(x) \approx \frac{4\eta - 1}{4x\sqrt{2 + \eta}} + O(x).$$

Likewise, near the singularity at $x = \infty$,

$$H(x) \approx \frac{\eta}{x^2} + O\left(\frac{1}{x^4}\right).$$

Therefore, if $\eta = 1/4$, then

$$H(x) = \frac{8x(x-3)(x+3)}{(4x^2 + 9)^{5/2}} \tag{35}$$

is bounded and the integral in (34) converges at both singular endpoints. Then from (33),

$$f_3(x) = 1 + \frac{9}{4x^2}$$

and

$$g_3(x) = \frac{-1}{4x^2}.$$

Hence the exponential powers in the asymptotic solution are computed from

$$\xi = \int_{a_j}^{x} f^{1/2}\, dx = \frac{\sqrt{4x^2+9}}{2} - \frac{3}{2}\log\left(\frac{3+\sqrt{4x^2+9}}{x}\right).$$

For the asymptotic solution, $\exp(\xi)$ can be cleaned up by identifying inverse functions and absorbing constants, so that the solution for Eq. (23) is approximated by a linear combination of the LG solutions

$$w_{3a} = \frac{Ax^2}{(4x^2+9)^{1/4}(3+\sqrt{4x^2+9})^{3/2}} \exp\left(\frac{\sqrt{4x^2+9}}{2}\right) \tag{36}$$

$$w_{3b} = \frac{B(3+\sqrt{4x^2+9})}{x(4x^2+9)^{1/4}} \exp\left(\frac{-\sqrt{4x^2+9}}{2}\right). \tag{37}$$

Fig. 3a graphs the predicted bound for the relative error from Eq. (5) against the relative exact error ($|w_c - w_{3a}|/w_c$), comparing exact and asymptotic solutions recessive near zero. The constant A in (37) is obtained here by matching w_c in the limit as $x \to 0$. Likewise Fig. 3b graphs the predicted error bound for w_{3b} against relative exact error ($|w_b - w_{3b}|/w_b$), comparing solutions recessive near infinity. This time, the constant B in (37) is obtained by matching w_b in the limit as $x \to \infty$. It is important to note that the error is truly bounded at $x = 0$, and not just asymptotic to the vertical axis. Observe that in both situations, the relative error is less than 10%, and the predicted upper bound is fairly sharp for all x.

Finally, for the case $\nu = 3/2$, the asymptotic solution for the modified Bessel equation (22) is formulated from (31) as

$$y_3 = \sqrt{x}(Aw_{3a} + Bw_{3b}). \tag{38}$$

Figure 3c graphs $\log(y_3(x))$ against $\log(y_r(x))$ for $0 < x < 40$, with constants in the asymptotic solution matching limiting values for recessive solutions near zero and infinity. The LG uniformly captures the nature and magnitude of the exact solution, as predicted by the error functions in Figs. 3a and 3b.

Without argument, the current partition produces a messier function than in the previous two formulations. However, the square and fourth roots of f_3 are still real and computable, and deliver a powerful approximation to the solution for all $x > 0$, even near the singularities at both zero and infinity!

5 Discussion

The LG is a powerful tool for approximating solutions to second-order differential equations in the vicinity of both ordinary and singular points, and can even return asymptotics that are equivalent to the exact solution, as observed in Euler's equation. With a proper partition of the coefficient function, it is capable of closely approximating exponential and oscillatory functions and provides reliable numerics in the neighborhood of irregular or regular singularities, as witnessed in Bessel's equation. However, it cannot approximate solutions that are neither exponential nor oscillatory in nature, such as the logarithmic functions that result from Euler's equation with repeated roots.

While the equations studied in this paper are solvable in elementary terms, using them to examine the LG facilitates a careful observation of how the asymptotics "imitate" a solution. The method was first applied to Euler's equation with what seemed to be a reasonable partition, obtaining a solution that also seemed reasonable. However, divergence of the error control function warned us that the approximation was unacceptable. Since Euler's equation is solvable in elementary terms, this result could be confirmed by comparison. Analysis of the error control for $h(x)$ led to the optimal partition, producing an error bound of zero. Likewise, with Bessel's equation, analysis of the error control integral facilitated solutions that are suitable for particular applications. Three different partitions produced approximations acceptable in distinct regions of the domain. Thus the error control function both directs and evaluates the asymptotic process.

Equations that occur in engineering and scientific applications are often not solvable in elementary terms and require numerical or asymptotic solutions near singular points. Numerical methods experience instabilities in singular regions, increasing both error and computation time. Under these circumstances, the LG may produce an elegant and more desirable solution. In this paper, we observed the power of Olver's error bound (see [8] Ch 6.2) for equations that could be solved analytically. When the LG is used to approximate a solution that cannot be obtained analytically, the error control function becomes an invaluable tool for assessing the relability of the approximation.

Many texts are still not clear in their exposition of the LG, particularly with respect to its ability to approximate functional values with predictable error in the neighborhood of singularities. Heading comments that "vagueness must be accepted as one of the inherent weaknesses of the [Liouville-Green] technique" [5]. By taking a look at the method in detail, I hope this paper has eliminated some of this "vagueness." In particular, the algorithm for obtaining the optimal partition, and resulting changes in the error bound should be useful for those applying the method.

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Fig. 1 LG approximates an exponential solution for Euler's equation only when the error control function is convergent. The exact and asymptotic solutions for $x^2z''(x) - 2z'(x) - z(x) = 0$ are graphed for $x > 0$. Initial partition of the coefficient function produces an unacceptable solution (15) away from initial conditions given at $x = 1$. Divergence of the integral in Eq. (16) predicts unbounded error for both large and small values of input.

Fig. 2 Two different LG partitions produce bounded error in distinct regions of the domain for Bessel's equation The modified Bessel equation $x^2y'' + xy' - (x^2 - 9/4)y = 0$ is converted to $y(x) = \phi(x)w(x)$, where $w'' = (1 + 2/x^2)w$. Two different partitions of the coefficient function produce acceptable asymptotics near one of the singularities, with unbounded error elsewhere. **(a)** The partition $f_1(x) = \frac{9}{4x^2}$ and $g_1(x) = 1 - \frac{1}{4x^2}$ produces bounded error near $x = 0$ that increases with x. The error bound ϵ_1 is graphed against relative error $|w_c - Ax^2|/w_c$, comparing analytic and asymptotic solutions recessive near zero. **(b)** The partition $f_2(x) = 1$ and $g_2(x) = \frac{2}{x^2}$ produces an acceptable solution for large values of x, but unbounded error near zero. The error bound ϵ_2 is graphed against relative error $|w_b - \exp(-x)|/w_b$, comparing analytic and asymptotic solutions recessive as $x \to \infty$.

Fig. 3 A comparison of predicted error bound and exact error validates Olver's error control function [8] for a final partition of Bessel's equation. The modified Bessel equation $x^2y'' + xy' - (x^2 - 9/4)y = 0$ is converted $y(x) = \phi(x)w(x)$, where $w'' = (1 + 2/x^2)w$. A final partition of the coefficient function produces two linearly independent asymptotic solutions, with one recessive at zero and the other recessive at infinity, to uniformly approximate the exact solution for all $x > 0$. **(a)** The predicted bound for relative error (5) is compared to relative exact error for solutions recessive at zero, Eqs. (27) and (37). **(b)** Likewise, the predicted error bound and relative error are compared for solutions recessive at infinity, Eqs. (24) and (37). **(c)** The analytic solution for Bessel's equation (31) is well approximated by the final LG asymptotic solution (38), as observed in the natural log of each function over the interval $(0, 40)$.

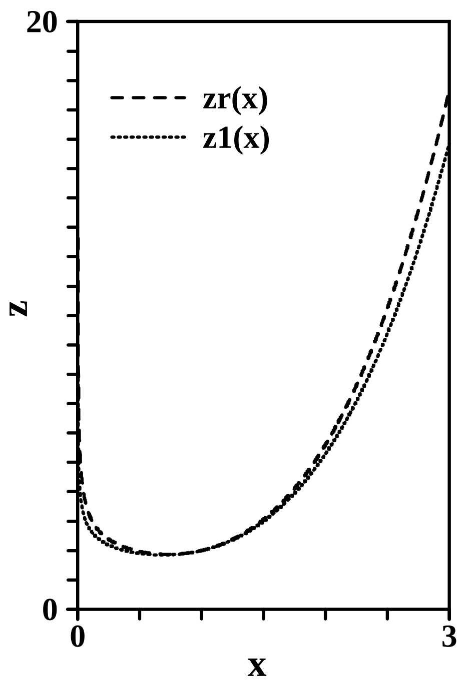


Figure 1:

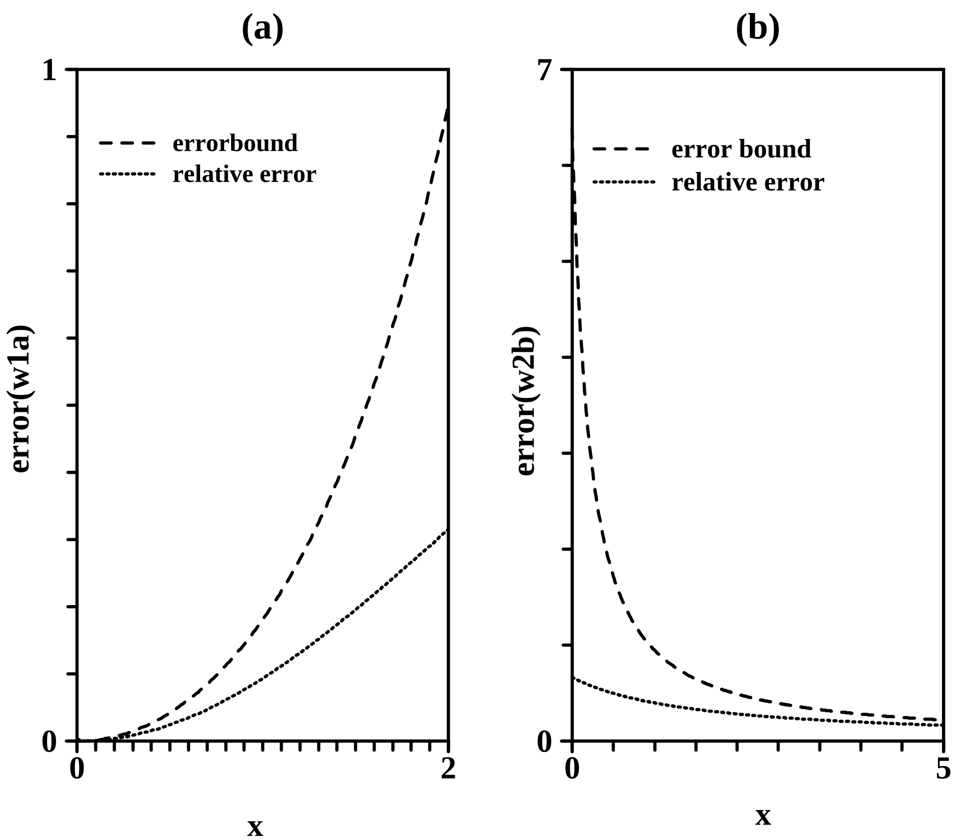
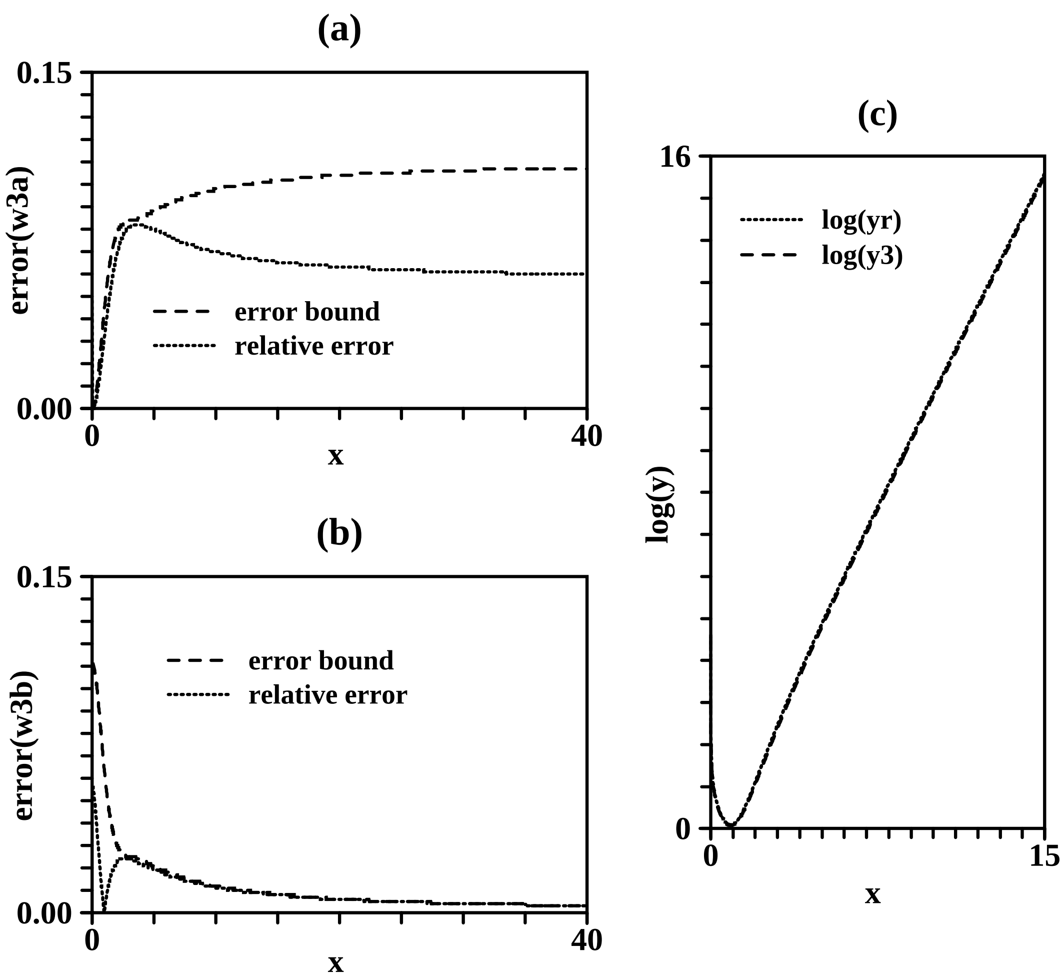

(a)
(b)
1
0
0
2
x
errorbound
relative error
error(w1a)
7
0
0
5
x
error bound
relative error
error(w2b)

Figure 2:


(a)
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error(w3a)
error bound
relative error
0.00
0
40
x
(b)
0.15
error(w3b)
error bound
relative error
0.00
0
40
x
(c)
16
log(yr)
log(y3)
log(y)
0
0
15
x

Figure 3: